The Generalized Principle of the Golden Section and its applications in mathematics, science, and engineering

A.P. Stakhov

International Club of the Golden Section, 6 McCreary Trail, Bolton, ON, Canada L7E 2C8

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Abstract

The “Dichotomy Principle” and the classical “Golden Section Principle” are two of the most important principles of Nature, Science and also Art. The Generalized Principle of the Golden Section that follows from studying the diagonal sums of the Pascal triangle is a sweeping generalization of these important principles. This underlies the foundation of “Harmony Mathematics”, a new proposed mathematical direction. Harmony Mathematics includes a number of new mathematical theories: an algorithmic measurement theory, a new number theory, a new theory of hyperbolic functions based on Fibonacci and Lucas numbers, and a theory of the Fibonacci and “Golden” matrices. These mathematical theories are the source of many new ideas in mathematics, philosophy, botanic and biology, electrical and computer science and engineering, communication systems, mathematical education as well as theoretical physics and physics of high energy particles.

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Algebra and Geometry have one and the same fate. The rather slow successes followed after the fast ones at the beginning. They left science at such step where it was still far from perfect. It happened, probably, because Mathematicians paid attention to the higher parts of the Analysis. They neglected the beginnings and did not wish to work on such field, which they finished with one time and left it behind.

Nikolay Lobachevsky

1. Introduction: “Fibonacci World” and “Harmony Mathematics”

To realize the real world as a whole, to see the reflection of one and the same essence, to see one reason in the set of all life forms, to see behind the natural laws one general law, from which all of them result, is the main tendency of human culture. The idea of Harmony that descended to us from ancient science is the “launching pad” for the
consideration of all natural laws from the general point of view. In the Ancient Greek’s philosophy, Harmony was opposite to Chaos and meant the organization of the Universe. The ingenious Russian philosopher Alexey Losev wrote: “From Plato’s point of view and generally from the view of all antique cosmology, the Universe by itself is the certain proportional whole that subordinate to the law of harmonic division, the Golden Section”.

There is presently a huge interest of modern science in the application of the Golden Section and Fibonacci numbers [1–16], which leads to supposition regarding the existence of what surrounds us, the so-called Fibonacci World, which is based upon the laws of the Golden Section and Fibonacci numbers. The “world” of animals and plants, the “world” of Man (including his morphological envelope and spiritual contents), and also the “world” of Music and Art, concerns the phenomena of the “Fibonacci world”. Modern research in crystallography [17], in astronomy [18], theoretical physics [19–29], and physics of the high energy particles [30–32] all confirm the fact that the “Physical World” is based on the Golden Section too.

For a simulation and mathematical description of the different “worlds”, which surround us, mathematicians have always created a corresponding mathematical theory. For example, for a simulation of mechanical and astronomical phenomena mathematicians created differential and integral calculus. For a description of the electromagnetic phenomena, Maxwell created the electromagnetism theory. The theory of probabilities was created for simulation of the “stochastic world.” These are but a few examples. In this connection there exists the idea of the creation of a new mathematical direction, Mathematics of Harmony, that would be intended for the mathematical simulation of those phenomena and processes of the objective world, for which Fibonacci numbers and the Golden Section are their objective essence (pine cone and pineapple, Shopen’s music, cordial activity of mammals, quasi-crystals, movement of the Solar system’s planets, the structure of genetic code, physics of the high energy particles, etc.). But this new mathematics can influence on the development of other areas of human culture, such as the basis of new computer projects and the reform of mathematical education.

These specific ideas became the contents of the article in [12], and also of the lecture “The Golden Section and Modern Harmony Mathematics” that was delivered at the 7th International conference “Fibonacci Numbers and Their Applications” (Austria, Graz, July, 1996) [13] and at the meeting of the Ukrainian Mathematical Society (Ukraine, Kiev, 1998) [14].

The goal of the present paper is to develop a new scientific principle called the Generalized Principle of the Golden Section. This one originates from the study of diagonal sums of the Pascal triangle and is a sweeping generalization of the Dichotomy Principle and the classical Golden Section Principle that descended to us from ancient science (Pythagoras, Plato, and Euclid). These are the principles, which have inspired our Harmony Mathematics, which forms the mathematical basis of various new ideas in mathematics, philosophy, general science (including computer science), engineering, botanic and biology, mathematical education, theoretical physics and the physics of the high energy particles.

2. The “Dichotomy Principle” and the classical “Golden Section Principle”

The remarkable book “Meta-language of the Living Nature” by the well known Russian architect Shevelev [33] is devoted to investigation of the most general principles that underlay the Living Nature. The “Dichotomy Principle” and the “Golden Section Principle” are the most important of them. The “Dichotomy Principle” is based on the following trivial property of the “binary” numbers:

\[ 2^n = 2^{n-1} + 2^{n-1}, \tag{1} \]

where \( n = 0, \pm 1, \pm 2, \pm 3, \ldots \)

For the case \( n = 0 \) we have

\[ 1 = 2^0 = 2^{-1} + 2^{-1}. \tag{2} \]

In the book [33] the following “dynamic” model of the “Dichotomy Principle” is given in the form of the infinite division of the “Unit” (“The Whole”) according to the “dichotomy” ratios (1), (2):

\[ 1 = 2^0 = 2^{-1} + 2^{-1} + 2^{-2} + 2^{-2} + 2^{-3} + 2^{-3} + 2^{-4} + 2^{-4} + 2^{-5} + 2^{-6} + \ldots = \sum_{i=1}^{\infty} 2^{-i} \]

\[ 1 = 2^0 = 2^{-1} + 2^{-2} + 2^{-3} + 2^{-4} + 2^{-5} + 2^{-6} + \ldots = \sum_{i=1}^{\infty} 2^{-i} \]

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The “Golden Section Principle” that came to us from Pythagoras, Plato, and Euclid is based on the following fundamental property that connects the adjacent powers of the Golden Ratio $\tau = \frac{1 + \sqrt{5}}{2}$:

$$\tau^n = \tau^{n-1} + \tau^{n-2},$$

where $n = 0, \pm 1, \pm 2, \pm 3, \ldots$

For the case $n = 0$ the identity (4) takes the following form:

$$1 = \tau^0 = \tau^{-1} + \tau^{-2}.$$  \hspace{1cm} (5)

Using the “golden” identities (4), (5) Shevelev developed in the book [33] the following “dynamic” model of the “Golden Section Principle”:

$$1 = \tau^0 = \tau^{-1} + \tau^{-2} = \tau^{-3} + \tau^{-4} + \tau^{-6} + \tau^{-7} + \tau^{-9} + \ldots = \sum_{i=1}^{\infty} \tau^{-(2i-1)}$$  \hspace{1cm} (6)

Notice that the “Dichotomy Principle” (3) and the “Golden Section Principle” (6) have a huge amount of applications in nature, science and mathematics (binary number system, numerical methods of the algebraic equation solutions, sell division and so on).

In particular, the Dichotomy Principle (3) underlies the binary number system:

$$A = \sum_{i} a_i 2^i$$  \hspace{1cm} (7)

where $a_i$ is the binary numeral 0 or 1; $2^i$ is the weight of the $i$th digit of the number system (7); $i = 0, \pm 1, \pm 2, \pm 3, \ldots$ The latter underlies modern computers and information technology.

Also the Golden Section Principle (6) is the basis of Bergman’s number system [4]:

$$A = \sum_{i} a_i \tau^i$$  \hspace{1cm} (8)

where $a_i$ is the binary numeral 0 or 1; $\tau^i$ is the weight of the $i$th digit of the number system (8); $\tau = \frac{1 + \sqrt{5}}{2}$ is the Golden Ratio; $i = 0, \pm 1, \pm 2, \pm 3, \ldots$.

3. The Generalized Principle of the Golden Section

3.1. A generalization of the Golden Section

The problem of the line division in extreme and middle ratio (the Golden Section) that came to us from the “Euclidean Elements” allows the following generalization. Let us give the integer non-negative number $p$ ($p = 0, 1, 2, 3, \ldots$) and divide the line $AB$ by the point $C$ in the following ratio (Fig. 1):

$$\frac{CB}{AC} = \left(\frac{AB}{CB}\right)^p.$$  \hspace{1cm} (9)

Let us denote by $x$ the ratio $AB:CB = x$; then according to (9) the ratio $CB:AC = x^p$. On the other hand, $AB = AC + CB$, from where the following algebraic equation follows:

$$x^{p+1} = x^p + 1.$$  \hspace{1cm} (10)

Let us denote by $\tau_p$ the positive root of the algebraic equation (10).

Eq. (10) describes an infinite number of the line segment $AB$ divisions in the ratio (9) because every $p$ “generates” its own variant of the division (9). For the case $p = 0$ we have: $\tau_p = 2$ and then the division (9) is reduced to the classical dichotomy (Fig. 1a). For the case $p = 1$ we have: $\tau_p = \tau = \frac{1 + \sqrt{5}}{2}$ (the Golden Ratio) and the division (9) coincides with the classical Golden Section (Fig. 1b). This fact is a cause why the division of the line segment in the ratio (9) was called the generalized Golden Section or the Golden $p$-Section [9] and the positive roots $\tau_p$ of the algebraic equation (10) were called the generalized Golden Proportions or the Golden $p$-Proportions [9].
Notice that there is a fundamental distinction between the division of the line segment in Fig. 1a and the rest divisions in Fig. 1b–e from the point of view “symmetry” and “asymmetry”. The division in Fig. 1a is based on the “Dichotomy Principle” and reflects the “Symmetry Principle”. The divisions in Fig. 1b–e are “asymmetric” divisions and reflect the “Asymmetry Principle.”

It follows from Eq. (10) the following fundamental identity that connects the adjacent powers of the Golden proportion $\tau_p$:

$$s_n^p = s_n^0 + s_n^{p-1} = \tau_p \times s_n^{p-1}. \quad (11)$$

where $n = 0, \pm 1, \pm 2, \pm 3, \ldots$

Notice that for the cases $p = 0$ and $p = 1$ the general identity (11) is reduced to the identities (1), (4) respectively.

3.2. The Generalized Principle of the Golden Section

And now let us divide all terms of the identity (11) by $s_n^p$. The following identity follows as a result of this division:

$$1 = s_n^0 = \tau_p^{-1} + \tau_p^{p-1}. \quad (12)$$

Using (11), (12) it is possible to construct the following “dynamic” model of the “Unit” decomposition according to the Golden $p$-Proportion:

$$1 = \tau_p^0 = \tau_p^{-1} + \tau_p^{(p+1)} = \tau_p^{(p+1)-1} + \tau_p^{2(p+1)} + \tau_p^{3(p+1)} + \ldots = \sum_{i=1}^{\infty} \tau_p^{-[(i-1)(p+1)-1]} \quad (13)$$

The main result of the above consideration is finding more general principle of the “Unit” division given by the following identity:

$$1 = \tau_p^{-1} + \tau_p^{(p+1)} = \sum_{i=1}^{\infty} \tau_p^{-[(i-1)(p+1)-1]}, \quad (14)$$

where $\tau_p$ is the Golden $p$-Proportion, $p \in \{0, 1, 2, 3, \ldots\}$.

It is clear that this general principle includes in itself the “Dichotomy Principle” (3) and the classical “Golden Section Principle” (6) as the special cases for $p = 0$ and $p = 1$.

3.3. The Generalized Principle of the Golden Section in modern philosophy

Recently the Byelorussian philosopher Eduardo Soroko developed very original approach to structural harmony of systems [5]. He used the generalized Golden Sections and formulated his famous “Law of Structural Harmony of Systems” as follows:

The Generalized Golden Sections are invariants, which allow natural systems in process of their self-organization to find harmonious structure, stationary regime of their existence, structural and functional stability.
According to Soroko’s opinion [5] this general law has the widest applications in Nature, Science and Art for modeling processes in self-organizing systems (substance structure, technology of structural-complicated products, ontogenetic structures of organism, human intellect and its creations). What is a principal peculiarity of Soroko’s Law for modern science of system harmony? Starting from Pythagoras the harmony researchers connected a harmony concept with the classical Golden Proportion \( \tau = \frac{1 + \sqrt{5}}{2} = 1.618 \) that is considered as the only and unique proportion of harmonious systems. Soroko’s Law asserts that there are an infinite number of the “harmonious” states of one and the same systems corresponding to the numbers \( \tau_p \) or to the opposite number \( \beta_p = \frac{1}{\tau_p} \), where \( p \) takes its value from the set of natural numbers.

Table 1 gives the values of the structural invariants \( \tau_p \) and \( \beta_p \), for the initial values of \( p \).

### Table 1

<table>
<thead>
<tr>
<th>( p )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \tau_p )</td>
<td>1.618</td>
<td>1.465</td>
<td>1.380</td>
<td>1.324</td>
<td>1.285</td>
<td>1.255</td>
<td>1.232</td>
</tr>
<tr>
<td>( \beta_p )</td>
<td>0.6180</td>
<td>0.6823</td>
<td>0.7245</td>
<td>0.7549</td>
<td>0.7781</td>
<td>0.7965</td>
<td>0.8117</td>
</tr>
</tbody>
</table>

4. The Generalized Principle of the Golden Section and Pascal triangle

Let us consider the following well-known mathematical formula called “Binomial formula”:

\[
(a + b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k.
\]

Numbers of combinations \( \binom{n}{k} \) in the formula (15) are named binomial coefficients.

The famous French mathematician Pascal offered the refined method of the binomial coefficients calculation based on their arrangement in the form of the special numerical table called Pascal triangle. Let us represent Pascal triangle in the form of the following numerical table:

<table>
<thead>
<tr>
<th>1</th>
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<tbody>
<tr>
<td>1</td>
<td>2</td>
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<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
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<tr>
<td>1</td>
<td>3</td>
<td>6</td>
<td>10</td>
<td>15</td>
<td>21</td>
<td>28</td>
<td>36</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>10</td>
<td>20</td>
<td>35</td>
<td>56</td>
<td>84</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>5</td>
<td>15</td>
<td>35</td>
<td>70</td>
<td>126</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>6</td>
<td>21</td>
<td>56</td>
<td>126</td>
<td></td>
<td></td>
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<tr>
<td>1</td>
<td>7</td>
<td>28</td>
<td>84</td>
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<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>8</td>
<td>36</td>
<td></td>
<td></td>
<td></td>
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<td></td>
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<tr>
<td>1</td>
<td>9</td>
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<tr>
<td>1</td>
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<td></td>
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<td></td>
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</tr>
</tbody>
</table>

Notice that the binomial coefficient \( \binom{n}{k} \) is on the intersection of the \( n \)th column (\( n = 0, 1, 2, 3, \ldots \)) and the \( k \)th row (\( k = 0, 1, 2, 3, \ldots \)) of the Pascal triangle. If we sum the binomial coefficients by columns we will get the “binary” sequence: 1, 2, 4, 8, 16, \ldots, \( 2^n \). In the combinatorial analysis this result is expressed in the form of the following elegant identity:

\[
2^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n}.
\]

Let us shift now every row of Pascal triangle by one column to the right with respect to the preceding column and consider the “deformed” Pascal triangle of the following kind:
If we sum the binomial coefficients of the “deformed” Pascal triangle we will come unexpectedly to the Fibonacci numbers: 1, 1, 2, 3, 5, 8, 13, 21, 34, ...!

If we shift every row of the initial Pascal triangle by \( p \) columns (\( p = 0, 1, 2, 3, \ldots \)) to the right with respect to the preceding column and then sum the binomial coefficients of the new “deformed” Pascal triangle by columns we will come to the numerical sequence that is expressed by the following recurrence relation:

\[
F_p(n) = F_p(n - 1) + F_p(n - p - 1) \quad \text{for} \quad n > p + 1;
\]

\[
F_p(1) = F_p(2) = \cdots = F_p(p + 1) = 1.
\]

Notice that the recurrence relation (16) for the initial terms (17) gives an infinite number of new numerical sequences. Moreover, the “binary” sequence 1, 2, 4, 8, 16, ... is the special case of this sequence for \( p = 0 \) and the classical Fibonacci numbers 1, 1, 2, 3, 5, 8, 13, ... are the special case of this sequence for \( p = 1 \)!

Let us consider now the ratio of the two adjacent Fibonacci \( p \)-numbers \( F_p(n)/F_p(n - 1) \) for the case \( n \rightarrow \infty \). It is proved [9] that this ratio strives to the Golden \( p \)-Proportion \( \tau_p \).

It is impossible to overestimate methodological importance of the deep mathematical connection of the Generalized Principle of the Golden Section given by (14) with Pascal triangle and binomial coefficients. It is clear that this connection can become the beginning for reappraisal of many branches of modern mathematics and theoretical physics where combinatorial relations play important role, in particular, probability theory and statistical laws.

5. A new measurement theory based on the Generalized Principle of the Golden Section

5.1. The first optimization problem in measurement theory

As is well known measurement theory has a long history. Its origin is connected to the “incommensurable segments” discovery made by Pythagoreans at investigation of the ratio of the square diagonal to its side. This discovery caused the first crisis in mathematics foundations and resulted to appearance of irrational numbers.

In 1202 the first optimization problem appeared in measurement theory. The famous Italian mathematician Fibonacci was the author of this problem. This problem is called the “problem of choosing the best system of standard weights” or Bashet–Mendellev’s problem (in the Russian mathematical literature [9]).

The essence of the problem consists in the following [9]. Let it be necessary to weigh any unknown weight \( Q \) in the range from 0 up to \( Q_{\text{max}} \) using \( n \) standard weights

\[
\{q_1, q_2, \ldots, q_n\},
\]

where \( q_1 = 1 \) is a measurement unit; \( q_i = k_1 \times q_1; k_i \) is any natural number.

It is clear that the maximum weight \( Q_{\text{max}} \) is equal to the sum of all standard weights, i.e.

\[
Q_{\text{max}} = q_1 + q_2 + \cdots + q_n = (k_1 + k_2 + \cdots + k_n)q_1.
\]

Then it appears the problem to find the optimal system of the standard weights, i.e. such standard weight system (18), which ensures the maximum value of \( Q_{\text{max}} \) given by (18) among all possible variants of (18). In this case we have to choose such variant of the standard weights (18) that it would be possible to compose any multiple by \( q_1 \) weight \( Q \) using the standard weights (18) taking each of them separately.

As is well known there are two variants of this problem [9]. For the former case we can place the standard weights only on the free cup of the balance, for the latter case we can place them on two cups of the balance.

The optimal solution for the former case is given by the “binary” system of standard weights, i.e.

\[
\{1, 2, 4, 8, 16, \ldots, 2^{n-1}\}.
\]

Notice that the measurement algorithm based on the “binary” system “generates” so-called “binary” measurement algorithm that is used widely in the measurement practice. Notice that the “binary” algorithm “generates” the “binary” number system that underlies modern computers and information technology.

5.2. The “Asymmetry Principle of Measurement”

The analysis of the “binary” algorithm by using the balance model (Fig. 2) allows to discover one surprising measurement property having a general character for all thinkable measuring that are reduced to the comparison of the measurable weight \( Q \) with any standard weights.

\[
\text{Algorithm} \quad \text{that is used widely in the measurement practice. Notice that the “binary” algorithm “generates” the “binary” number system that underlies modern computers and information technology.}
\]
Let us consider very carefully the process of weighing the weight $Q$ on the balance using the “binary” system of standard weights. At the first step of the “binary algorithm” the largest standard weight $2^{n-1}$ places on the free cup of the balance (Fig. 2a). After the first step one may appear two situations for the cases $2^{n-1} < Q$ (Fig. 2a) and $2^{n-1} \geq Q$ (Fig. 2b). In the former case (Fig. 2a) the second step is to place the next large standard weight $2^{n-2}$ to the free cup of the balance. In the latter case the “weigher” should perform two operations, i.e. remove the previous standard weight $2^{n-1}$ from the free cup of the balance (Fig. 2b) so that the balance returns to the initial position (Fig. 2c). Then the next standard weight $2^{n-2}$ places on the free cup of the balance (Fig. 2c).

One can readily see that the both considered cases differ between themselves by their “complexity”. In fact, in the former case the “weigher” performs only one operation, i.e. he places the next standard weight $2^{n-2}$ on the free cup of the balance. In the latter case the “weigher’s” actions are determined by two reasons. First he has to remove the previous standard weight $2^{n-1}$ from the free cup of the balance and then to wait when the balance returns to the initial position. After the returning the balance to the initial position (Fig. 2c) the “weigher” places the next standard weight $2^{n-2}$ on the free cup of the balance (Fig. 2c).

The discovered property of measurement was called the “Asymmetry Principle of Measurement” [9].

5.3. Fibonacci’s measurement algorithm

Application of the “Asymmetry Principle of Measurement” to Bashet–Mendeleev’s problem had led to the unexpected result. It is proved [9] that for this case the optimal solution of the problem is reduced to the following system of standard weights:

$$\{F_p(1), F_p(2), \ldots, F_p(l), \ldots, F_p(n)\}$$

(21)

where $F_p(i)$ is the Fibonacci $p$-number given by (16), (17).

It is proved [9] that Fibonacci’s measurement algorithm based on (21) is reduced to the division of line segment in the ratio (16). It means that Fibonacci’s measurement algorithm is based on the Generalized Principle of the Golden Section!

It is important to emphasize that Fibonacci’s measurement algorithm is natural consequence of the “Asymmetry Principle of Measurement” (Fig. 2). This conclusion about the deep connection between the Generalized Principle of the Golden Section and the “Asymmetry Principle” has methodological importance. One may assume that the Generalized Principle of the Golden Section appears always when we find “asymmetry” in the study of natural phenomenon.

Fig. 2. The “Asymmetry Principle of Measurement”.
6. The “Asymmetry Principle” of the living nature

6.1. “Asymmetry” of “rabbits reproduction”

As is well known the classical Fibonacci numbers 1, 1, 2, 3, 5, 8, 13, 21, 34, ... were introduced by the famous Italian mathematician Leonardo Pisano Fibonacci in the 13 century at the solution of the well-known “rabbits reproduction” problem. This problem and its solution (Fibonacci numbers) have two important corollaries for development of modern science. First of all the “rabbits reproduction” problem gave an origin of the mathematical theory of biological populations [34]. Second, the Fibonacci recurrence relation gave an origin of the recurrence relation method that is one of the most important methods of combinatorial analysis [1–3].

Let us remind ourselves that the “Law of rabbit reproduction” is reduced to the following rule. Each “mature rabbit couple” $A$ births a “newborn rabbit couple” $B$ during one month. The “newborn rabbit couple” matures during one month and one more in one month starts to bring one rabbit couple. Thus, “maturing” the newborn rabbits, that is, their transformation into mature couple is realized in 1 month. We can model a process of “rabbits reproduction” by using two passages:

\[ A \rightarrow AB \]  
\[ B \rightarrow A \]  

Notice that the passage (22) models a process of the “birth” of the “newborn rabbit couple” $B$ and the passage (23) models a process of “maturing” the “newborn rabbit couple” $B$. The passage (22) reflects “asymmetry” of “rabbits reproduction” because the “mature rabbit couple” $A$ turns into the two non-identical couples, the “mature rabbit couple” $A$ and the “newborn rabbit couple” $B$.

6.2. The generalized “Asymmetry Principle” of the living nature

Using the model of “rabbits reproduction” one may generalize the “Asymmetry Principle” of the Living Nature. To this end we will take the non-negative number $p \geq 0$ and consider the following problem:

“Let us suppose that in the enclosed place there is a couple of rabbits (female and male) in the first day of January. This rabbit couple reproduces a new rabbit couple in the first day of February and then in the first day of each next month. Each newborn rabbit’s couple becomes mature in $p$ months and then gives a life to the new rabbit’s couple each month after. There is a question: how much rabbit’s couples will be in the enclosed place in one year, that is, in 12 months from the beginning of reproduction?”

It is clear that for the case $p = 1$ the generalized variant of the “rabbit reproduction” problem coincides with the classical “rabbits reproduction” problem [1–3].

Note that the case $p = 0$ corresponds to that idealized situation when “rabbits” become “mature” right away after birth. One may model this case using the passage:

\[ A \rightarrow AA \]  

It is clear that that the passage (24) reflects “symmetry” of “rabbits reproduction” when the “mature rabbit couple” $A$ turns into the two identical “mature rabbit couple” $A$. It is easy to show that for this case the “rabbits” are reproduced according to the “Dichotomy principle”, that is, the “rabbits” double each month: 1, 2, 4, 8, 16, 32, ...

Let us consider now the case $p > 0$. Let us analyze more in detail the formulated above problem taking into consideration the new conditions of “rabbit reproduction”. It is clear that “reproduction process” is described by more complex system of “passages” describing the “reproduction process”. Really, let $A$ and $B$ be the couples of “mature” and “newborn” rabbits respectively. Then the passage (22) models a process of monthly appearance of the “newborn couple” $B$ from each “mature couple” $A$.

Let us consider now a process of transformation of the “newborn couple” $B$ into the “mature couple” $A$. It is evident that during “maturing” the “newborn couple” $B$ passes through intermediate stages corresponding to each month:

\[ B \rightarrow B_1 \]
\[ B_1 \rightarrow B_2 \]
\[ B_2 \rightarrow B_3 \]
\[ \ldots \ldots \]
\[ B_{p-1} \rightarrow A \]
For example, for the case $p = 2$ a process of transformation of the “newborn couple” into the “mature couple” is described by the following system of “passages”:

$$\begin{align*}
B & \rightarrow B_1 \\
B_1 & \rightarrow A
\end{align*}$$

Then, taking into consideration (22), (26), (27) the process of “rabbit reproduction” for the case $p = 2$ can be represented with help of Table 2.

<table>
<thead>
<tr>
<th>Date</th>
<th>Rabbits couples</th>
<th>$A$</th>
<th>$B$</th>
<th>$B_1$</th>
<th>$A + B + B_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>January, 1</td>
<td>$A$</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>February, 1</td>
<td>$AB$</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>March, 1</td>
<td>$A \ B_1$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>April, 1</td>
<td>$AB \ B_1 A$</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>May, 1</td>
<td>$AB \ B_1 A A B$</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td>6</td>
</tr>
<tr>
<td>June, 1</td>
<td>$AB \ B_1 A A A B \ B_1$</td>
<td>4</td>
<td>3</td>
<td>2</td>
<td>9</td>
</tr>
<tr>
<td>July, 1</td>
<td>$AB \ B_1 A A B A B \ B_1 A \ AB \ B_1 A$</td>
<td>6</td>
<td>4</td>
<td>3</td>
<td>13</td>
</tr>
<tr>
<td>August, 1</td>
<td>$AB \ B_1 A A B A B \ B_1 A B \ B_1 A A B A A B$</td>
<td>9</td>
<td>6</td>
<td>4</td>
<td>19</td>
</tr>
</tbody>
</table>

shows that they are subordinated to one and the same regularity: each number of the sequence is equal to the sum of the preceding number and the number distant from the latter in two positions. But we know the Fibonacci $2$-numbers given by the recurrence relation (16) are subordinated to that regularity!

If we carry out similar considerations for the general case of $p$ we will come to conclusion that the Fibonacci $p$-numbers given by the recurrence relation (16) would be a solution of the generalized variant of the “rabbit reproduction” problem! And they reflect the “Generalized Asymmetry Principle” of the Living Nature.

At first sight, the above formulation of generalized problem of “rabbit reproduction” has no real “physical” sense. But we will not hurry with conclusions! The paper [34] is devoted to application of the generalized Fibonacci numbers for modeling of biological cell growth. In the paper it is affirmed that “in kinetic analysis of cell growth, the assumption is usually made that cell division yields two daughter cells symmetrically. The essence of the semi-conservative replication of chromosomal DNA implies complete identity between daughter cells. Nonetheless, in bacteria, insects, nematodes, and plants, cell division is regularly asymmetric, with spatial and functional differences between the two products of division… Mechanism of asymmetric division include cytoplasmic and membrane localization of specific proteins or of messenger RNA, differential methylation of the two strands of DNA in a chromosome, asymmetric segregation of centrioles and mitochondria, and bipolar differences in the spindle apparatus in mitosis”. In [34] it is analyzed the models of cell growth based on the Fibonacci 2- and 3-numbers.

In the summery the authors of [34] made the following conclusion: “Binary cell division is regularly asymmetric in most species. Growth by asymmetric binary division may be represented by the generalized Fibonacci equation … Our models, for the first time at the single cell level, provide rational bases for the occurrence of Fibonacci and other recursive phyllotaxis and patterning in biology, founded on the occurrence of regular asymmetry of binary division”.

Thus, the results of the paper [34] show that the world of biology is based on the “Generalized Principle of the Golden Section”!
7. A new theory of real numbers based on the generalized principle of the Golden Section

7.1. Euclidean definition of natural numbers

A number is one of the most important mathematical concepts. During many millennia this concept is widened and made more precise. A discovery of irrational numbers led to the concept of real numbers that include natural numbers, rational and irrational numbers. Further development of the number concept is connected to introduction of complex numbers and their generalizations, the hypercomplex numbers. Let us start from the natural numbers concept that underlies the basis of number theory. We know from the "Euclidean Elements" the following "geometric approach" to the natural number definition.

Let \( S = \{1, 1, 1, \ldots\} \) (28)

be an infinite set of geometric segments called "Monads" or "Units". Then according to Euclid we can define a natural number \( N \) as a sum of "Monads":

\[
N = 1 + 1 + 1 + \cdots + 1 \text{ (}N\text{ times)}. \tag{29}
\]

Despite of limiting simplicity of such definition, it has played a large role in mathematics and underlies many useful mathematical concepts, for example, concepts of prime and composite numbers, and also concept of divisibility, one of the main concepts of number theory.

7.2. A constructive approach to the number concept

It is known so-called "constructive approach" to the number definition based on the "Dichotomy Principle". According to the "constructive approach" any "constructive" real number \( A \) is some mathematical object given by the mathematical formula (7).

The number definition given by (7) has the following geometric interpretation. Let \( B = \{2^n\} \) (30)

be the set of the geometric line segments of the length \( 2^n(n = 0, \pm1, \pm2, \pm3,\ldots) \). Then all geometric line segments that can be represented as the final sum (7) can be defined as constructive real numbers.

It is clear that the definition (7) selects from the set of all real numbers only some part of real numbers that can be represented in the form (7). All the rest real numbers that cannot be represented in the form of the sum (7) are called non-constructive. It is clear that all irrational numbers, in particular, the mathematical constants \( \pi, e \), the Golden Section refer to the non-constructive numbers. But in the framework of the definition (7) we should refer to the non-constructive one’s and some rational numbers (for example, 2/3, 3/7 and so on) that are known as periodical fractions that cannot be represented in the form of final sum (7).

Notice that although the number definition (7) restricts considerably the set of real numbers this fact does not diminish its importance from the "practical", computational point of view. It is easy to prove that any non-constructive real number can be represented in the form (7) approximately, here the approximation error \( \Delta \) would be increased with no limit if we increase a number of terms in (7). However, \( \Delta \neq 0 \) for all the non-constructive real numbers. By essence, we use in modern computers only the constructive real numbers given by (7).

7.3. Newton’s definition of real numbers

Within many millennia the mathematicians developed and made more precise the concept of number. In the 17th century during origin of modern science and mathematics the new methods of studying the "continuous" processes (integration and differentiation) are developed and the concept of real numbers again goes out on the foreground. Most clearly the new definition of this concept is given by Isaac Newton, one of the mathematical analysis founders, in his "Arithmetica Universalis" (1707):

We understand as numbers not so much set of units, how many the abstract ratio of any magnitude to other one of the same kind taken by us for the unit.

This formulation gives us the uniform definition of all real numbers, rational and irrational. If we consider now the "Euclidean definition" (29) from the point of "Newton’s definition" then the "monad" plays a role of the Unit in it. In the "binary" notation (30) the number 2, the radix of the number system, plays a role of the Unit.
7.4. A new constructive definition of real number

Let us consider the infinite set of the “standard segments” based on the Golden \( p \)-Ratio \( \tau_p \):

\[
G_p = \{ \tau_p^n \};
\]

(31)

where \( n = 0, \pm 1, \pm 2, \pm 3, \ldots \); \( \tau_p^n \) are the Golden \( p \)-Ratios powers connected among themselves by the identity (11).

The set (31) “generates” the following constructive method of the real number \( A \) representation called the code of the Golden \( p \)-Proportion:

\[
A = \sum_i a_i \tau_p^i,
\]

(32)

where \( a_i \in \{0, 1\} \) and \( i = 0, \pm 1, \pm 2, \pm 3, \ldots \).

Notice that the positional number systems (32) were introduced by the present paper author in 1980 in the paper [35] and called the Codes of the Golden \( p \)-Proportion. A theory of these number systems was developed in author’s book [11].

Let us consider the partial cases of the number representation (32). It is clear that for the case \( p = 0 \) the formula (32) is reduced to (7). Finally, let us consider the case \( p \to \infty \). For this case it is possible to show, that \( \tau_p \to 1 \); it means that the positional representation (32) is reduced to the Euclidean definition (29).

Notice that for the case \( p > 0 \) the radix \( \tau_p \) of the positional number system (32) is irrational number. It means that we have came to the number systems with irrational radices that a principally new class of the positional number systems. Notice that for the case \( p = 1 \) the number system (32) is reduced to the Bergman’s number system (8) that had been introduced by the American mathematician George Bergman in 1957 [4]. Notice that Bergman’s number system (8) is the first number system with irrational radix in the history of mathematics.

It follows from the given consideration that the positional representation (32) is very wide generalization of the classical binary number system (7), Bergman’s number system (8) and the Euclidean definition (29) that are partial cases of the general representation (32).

Possibly the number system with irrational radix (8) developed by George Bergman in 1957 and its generalization given by (32) are the most important mathematical discoveries in the field of number systems after discovery of positional principle of number representation (Babylon, 2000 B.C.) and decimal number system (India, 5th century).

7.5. Some properties of the number systems with irrational radices

Note that the expression (32) divides the set of real numbers into two non-overlapping subsets, the “constructive” real numbers that can be represented in the form of the final sum (32) and all the rest real numbers that cannot be represented in the form of the sum (32) and are called the “non-constructive” real numbers. Such approach to real numbers is distinguished radically from the classical approach when the set of real numbers is divided into rational and irrational numbers.

Really, all powers of the Golden \( p \)-Proportion of the kind \( \tau_p^i \) \((i = 0, \pm 1, \pm 2, \pm 3, \ldots)\) that are irrational numbers can be represented in the form (32), that is, they refer to the subset of the “constructive” numbers. For example,

\[
\begin{align*}
\tau_p^1 &= 10, & \tau_p^{-1} &= 0.1 \\
\tau_p^2 &= 100, & \tau_p^{-2} &= 0.01 \\
\tau_p^3 &= 1000, & \tau_p^{-3} &= 0.001.
\end{align*}
\]

It follows from the definition (32) that all real numbers that are the sums of the Golden \( p \)-Proportion powers are “constructive” numbers of the kind (32). For example, according to (32) the real number \( A = \tau_p^2 + \tau_p^{-1} + \tau_p^{-3} \) can be represented as the following binary code combination:

\[
A = 100.1101.
\]

Notice that a possibility of representation of some irrational numbers (the powers of the Golden \( p \)-Proportion and their sums) in the form of the final totality of bits is the first unusual property of the number systems (32).

7.6. Representation of natural numbers

Let us consider the representation of natural numbers in the form (32):

\[
N = \sum_i a_i \tau_p^i
\]

(33)

where \( N \) is some natural number, \( \tau_p \) is the radix of number system (33), \( a_i \in \{0, 1\} \), \( i = 0, \pm 1, \pm 2, \pm 3, \ldots \).
It is proved in [11] that all representations of the kind (33) are finite, that is, for the given $p \geq 0$ any sum (33) consists of an finite number of terms. For example, for the case $p = 1$ (Bergman’s number system) the initial terms of natural series can be represented as follows:

$$1 = 1, 0; 2 = 10, 01; 3 = 100, 01; 4 = 101, 01; 5 = 1000, 1001; 6 = 1010, 0001; 7 = 10000, 0001.$$ 

This result has a great importance for mathematics and general science. As all natural numbers can be represented in the form (33) it means that one may formulate new scientific doctrine “Everything is the Golden p-Proportion” instead the Pythagorian doctrine “Everything is a number”.

7.7. Z-property of natural numbers

Let us prove that the new definition of real number (32), (33) based on the Generalized Principle of the Golden Section can become a source of new number-theoretical results. The Z-property [14] of natural numbers is one of them.

For the proof of this property let us represent some natural number $N$ in Bergman’s number system:

$$N = \sum_i a_i \tau^i$$

where $\tau = \frac{1 + \sqrt{5}}{2} = 1.618$ and $i = 0, \pm 1, \pm 2, \pm 3, \ldots$

The expression (34) is called the $\tau$-code of natural number $N$.

It is well known the following formula in the Fibonacci numbers theory [1–3]:

$$\tau^i = L_i + F_i \sqrt{5}$$

where $F_i$ and $L_i$ are Fibonacci and Lucas numbers, $\tau$ is the Golden Proportion and $i = 0, \pm 1, \pm 2, \pm 3, \ldots$ It is considered that the formula (35) had been deduced in the 19 century by the French mathematician Binet.

Using formula (35) it is easy to prove the following theorem [14].

**Theorem 1 (Z-property of natural numbers).** If we represent some natural number $N$ in Bergman’s number system (34) and then replace every power of the Golden Ratio $\tau^i$ in the expression (34) by the Fibonacci number $F_i$, where the index $i$ takes its values from the set $\{0, \pm 1, \pm 2, \pm 3, \ldots\}$, then the sum arising as result of such replacing is equal to 0 identically independently on the initial natural number $N$, that is,

$$\sum_i a_i F_i = 0.$$ 

In [14] it is proved the following theorem that is a generalization of Theorem 1.

**Theorem 2 (Zp-property of natural numbers).** If we represent some natural number $N$ in the Golden p-Ratio code $N = \sum_i a_i \tau^i_p$, where $p > 0$ and $i = 0, \pm 1, \pm 2, \pm 3, \ldots$, and replace each power of the Golden p-Ratio in it with the corresponding Fibonacci p-number $F_p(i)$ then the sum arising at this replacement is identically equal to 0 independently on the initial natural number $N$, that is,

$$\sum_i a_i F_p(i) = 0.$$ 

Note that the properties given by Theorems 1 and 2 are valid only for natural numbers! It means that our investigations have led us to discovery of the new property of natural numbers called Z- or Zp-property (from the word “Zero”). And this unusual property could be discovered in mathematics only after discovery of number systems with irrational radices given by (8) and (32).

8. Fibonacci and “Golden” matrices

8.1. Q-matrix

In the last decades the theory of the Fibonacci numbers was supplemented by the theory of so-called Q-matrix [3]. The latter presents by itself the $2 \times 2$ matrix of the following form:

$$Q = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}. \quad (36)$$

It is easy to calculate the Q-matrix determinant, which equals to $-1$. 

In the paper [36] devoted to the memory of Verner E. Hoggat, the creator of the Fibonacci Association, it was stated the history of the \( Q \)-matrix, given an extensive bibliography on the \( Q \)-matrix and emphasized Hoggat’s contribution in development of the \( Q \)-matrix theory. Although the name of the “\( Q \)-matrix” was introduced before Verner E. Hoggat, just starting from Hoggat’s papers the idea of the \( Q \)-matrix “caught on like wildfire among Fibonacci enthusiasts. Numerous papers have appeared in Fibonacci Quarterly authored by Hoggatt and/or his students and other collaborators where the \( Q \)-matrix method became a central tool in the analysis of Fibonacci properties” [36].

The \( Q \)-matrix (36) has a number of remarkable properties concerning to the Fibonacci numbers. In particular, the \( n \)th power of the \( Q \)-matrix is the \( 2 \times 2 \) matrix of the following form:

\[
Q^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}.
\tag{37}
\]

It is easy to prove the following identity for the matrix \( Q^n \):

\[
Q^n = Q^{n-1} + Q^{n-2}
\tag{38}
\]

that is similar to the recurrence relation for Fibonacci numbers \( F_n = F_{n-1} + F_{n-2} \).

It is proved [2] that the determinant of the matrix (37) is equal to

\[
\text{Det } Q^n = F_{n+1} \times F_{n-1} - F_n^2 = (-1)^n,
\tag{39}
\]

where \( F_{n-1}, F_n, F_{n+1} \) are the adjacent Fibonacci numbers.

Notice that the formula (39) following from (37) is one of the most important theorems of the Fibonacci numbers theory [1–3].

8.2. Fibonacci \( Q_p \)-matrices

Using the idea of the \( Q \)-matrix (37) the following generalization of the \( Q \)-matrix had been introduced in [37]:

\[
Q_p = \begin{pmatrix} 1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.
\tag{40}
\]

Such matrices were called in [37] Fibonacci \( Q_p \)-matrices.

Let us analyze the matrix (40). First of all we notice that all elements of the \( Q_p \)-matrix (40) are equal to 0 or 1; here the first column begins from 1 and ends by 1 but all its rest elements are equal to 0, the last row of the matrix (40) begins from 1 and all the rest elements are equal to 0. The rest part of the matrix (40) (without the first column and the last row) is an identity \((p \times p)\)-matrix. For the cases \( p = 0, 1, 2, 3, 4 \) the corresponding \( Q_p \)-matrices have the following form respectively:

\[
Q_0 = (1); \quad Q_1 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = Q; \quad Q_2 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}; \quad Q_3 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}; \quad Q_4 = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}.
\]

Notice that for the case \( p = 1 \) the \( Q_1 \)-matrix coincides with the classical \( Q \)-matrix (36). Notice also that the \( Q_p \)-matrices have exceptional regularity. For example, the \( Q_{p-1} \)-matrix \((p = 1, 2, 3, \ldots)\) can be obtained from the \( Q_p \)-matrix by means of crossing out the last column and the next to the last row in the latter. It means that each \( Q_p \)-matrix as if includes in itself all preceding \( Q_p \)-matrices and is contained into all the next \( Q_p \)-matrices.
In [37] it is proved the following theorems for the Fibonacci $Q_p$-matrices.

**Theorem 3.** For the given $p = 0, 1, 2, 3, \ldots$ and $n = 0, \pm 1, \pm 2, \pm 3, \ldots$ we have the following property for the $n$th power of the $Q_p$-matrix:

$$Q_p^n = \begin{pmatrix}
F_p(n+1) & F_p(n) & \cdots & F_p(n-p+2) & F_p(n-p+1) \\
F_p(n+p) & F_p(n) & \cdots & F_p(n-2p+2) & F_p(n-2p+1) \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
F_p(n-1) & F_p(n-2) & \cdots & F_p(n-p) & F_p(n-p-1) \\
F_p(n) & F_p(n-1) & \cdots & F_p(n-p+1) & F_p(n-p) \\
\end{pmatrix}$$  (41)

Notice that the Fibonacci $p$-numbers (16), (17) are the entries of the matrix (41). It means that matrix (41) has direct relation to Pascal triangle!

**Theorem 4**

$$\det Q_p^n = (-1)^{pn},$$  (42)
i.e., $p = 0, 1, 2, 3, \ldots; n = 0, \pm 1, \pm 2, \pm 3, \ldots$

It is easy to prove [37] the following remarkable identity for the matrix (41):

$$Q_p^n = Q_p^{n-1} + Q_p^{n-p-1}$$  (43)

that is similar to the identity (16).

Notice that the formulas (36)-(39) are the partial cases of the more general expressions (40)-(43). Notice also that the expressions (38), (43) are similar to the expressions (4), (9). It means that the Generalized Principle of the Golden Section underlies the $Q$- and $Q_p$-matrices.

### 8.3. Hyperbolic Fibonacci and Lucas functions

In [38–40] a new class of hyperbolic functions based on Fibonacci and Lucas numbers had been introduced. Let us consider so-called symmetric hyperbolic Fibonacci functions introduced in [40]:

**Symmetric hyperbolic Fibonacci sine:**

$$sFs(x) = \frac{x^2 + x^{-2}}{\sqrt{5}}.$$  (44)

**Symmetric hyperbolic Fibonacci cosine:**

$$cFs(x) = \frac{x^2 - x^{-2}}{\sqrt{5}}.$$  (45)

The hyperbolic Fibonacci and Lucas functions [38–40] are a sweeping generalization of Fibonacci and Lucas sequences and coincide with the hyperbolic Fibonacci and Lucas functions for the discrete values of the continues variable $x$, that is, in the points $x = 0, \pm 1, \pm 2, \pm 3, \ldots$. In particular, the symmetric hyperbolic Fibonacci functions are connected to Fibonacci numbers $F_n$ with the following correlation:

$$F_n = \begin{cases}
sFs(n) & \text{for } n = 2k \\
cFs(n) & \text{for } n = 2k + 1. \\
\end{cases}$$  (46)

In [40] it is proved many remarkable properties of the symmetric Fibonacci and Lucas functions. In particular, the following identities are valid for the functions (44), (45):

$$[sFs(x)]^2 - cFs(x + 1)cFs(x - 1) = -1$$  (47)

$$[cFs(x)]^2 - sFs(x + 1)sFs(x - 1) = 1.$$  (48)

Notice that the identities (47), (48) are a generalization of the fundamental identity (39) that connects three adjacent Fibonacci numbers.
8.4. The “Golden” matrices

Let us represent the matrix (37) in the form of the two matrices given for the even \((n = 2k)\) and odd \((n = 2k + 1)\) values of the index \(n\):

\[
Q^{2k} = \begin{pmatrix}
F_{2k+1} & F_{2k} \\
F_{2k} & F_{2k-1}
\end{pmatrix},
\]

(49)

\[
Q^{2k+1} = \begin{pmatrix}
F_{2k+2} & F_{2k+1} \\
F_{2k+1} & F_{2k}
\end{pmatrix},
\]

(50)

Using the correlation (46) it is possible to represent the matrices (49), (50) in the terms of the symmetric hyperbolic Fibonacci functions (44), (45):

\[
Q^{2k} = \begin{pmatrix}
\cFs(2k + 1) & \sFs(2k) \\
\sFs(2k) & \cFs(2k - 1)
\end{pmatrix},
\]

(51)

\[
Q^{2k+1} = \begin{pmatrix}
\sFs(2k + 2) & \cFs(2k + 1) \\
\cFs(2k + 1) & \sFs(2k)
\end{pmatrix},
\]

(52)

where \(k\) is the discrete variable, \(k = 0, \pm 1, \pm 2, \pm 3, \ldots\)

And now we will replace the discrete variable \(k\) in the matrices (51), (52) by the continues variable \(x\):

\[
Q^x = \begin{pmatrix}
\cFs(2x + 1) & \sFs(2x) \\
\sFs(2x) & \cFs(2x - 1)
\end{pmatrix},
\]

(53)

\[
Q^{x+1} = \begin{pmatrix}
\sFs(2x + 2) & \cFs(2x + 1) \\
\cFs(2x + 1) & \sFs(2x)
\end{pmatrix},
\]

(54)

It is clear that the matrices (53), (54) are a generalization of the \(Q\)-matrix (37) for continues domain. They have a few unusual mathematical properties. For example, for the case \(x = \frac{1}{4}\) the matrix (53) takes the following form:

\[
Q^\frac{1}{4} = \sqrt{Q} = \begin{pmatrix}
\cFs(\frac{1}{2}) & \sFs(\frac{1}{2}) \\
\sFs(\frac{1}{2}) & \cFs(-\frac{1}{2})
\end{pmatrix}.
\]

(55)

It is impossible to imagine that means the “root square from the \(Q\)-matrix” but this “Fibonacci’s fantasy” follows from the expression (55).

And if we calculate the determinants of the matrices (53), (54) then using (47), (48) we will come to one more “fantastic” results that are valid for any value of the continues variable \(x\):

\[
\text{Det} Q^x = 1,
\]

(56)

\[
\text{Det} Q^{x+1} = -1.
\]

(57)

Thus, Fibonacci numbers, Fibonacci \(p\)-numbers, and the hyperbolic Fibonacci functions have led us to very unusual matrices given by (37), (41), (53), (54). Their extraordinary nature consists in the fact that according to (42), (56), (57) their determinants are equal 1 or \(-1\) independently on the continues variable \(x\)!

9. The generalized Principle of the Golden Section in computer engineering

9.1. Fibonacci codes and arithmetic

Fibonacci’s algorithms considered above are isomorphic to the following positional representation of natural number \(N\) [9]:

\[
N = a_n F_p(n) + a_{n-1} F_p(n - 1) + \cdots + a_1 F_p(1) + \cdots + a_i F_p(i),
\]

where \(a_i \in \{0, 1\}\) is the binary numeral of the \(i\)th digit of the code (58); \(n\) is the digit number of the code (58); \(F_p(i)\) is the \(i\)th digit weight calculated in accordance with the recurrent relation (14) at the initial conditions (15).
A positional representation of the natural number \( N \) in the form (58) is called the Fibonacci \( p \)-code. The abridged notation of the Fibonacci \( p \)-code (58) has the following form:

\[
N = a_n a_{n-1} \cdots a_i \cdots a_1.
\]  

(59)

Notice that the notion of the Fibonacci \( p \)-code includes an infinite amount of different methods of the binary representations as every number \( p \) "generates" its own Fibonacci \( p \)-code \((p = 0, 1, 2, 3, \ldots)\).

Let us consider the partial cases of the Fibonacci \( p \)-code (58). For the case \( p = 0 \) the Fibonacci \( p \)-numbers given by (14), (15) are reduced to the binary numbers: 1, 2, 4, 8, 16, 32, 64, 128, \ldots, \( 2^{i-1}, \ldots \), that is,

\[
F_0(i) = 2^{-i}.
\]  

(60)

Substituting (60) into the formula (58) we have:

\[
N = a_n 2^{n-1} + a_{n-1} 2^{n-2} + \cdots + a_i 2^{i-1} + \cdots + a_1 2^0.
\]  

(61)

It means that for the case \( p = 0 \) the Fibonacci \( p \)-code (58) is reduced to the classical binary representation of natural numbers.

Let \( p = 1 \). For this case the Fibonacci \( p \)-numbers are reduced to the classical Fibonacci numbers: 1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots, \( F_n \). It is clear that the Fibonacci \( p \)-code (58) for this case is reduced to so-called Zeckendorf’s representation [2]:

\[
N = a_n F_n + a_{n-1} F_{n-1} + \cdots + a_1 F_1 + \cdots + a_1 F_1
\]  

(62)

Let us consider now the partial case \( p = \infty \). In this case every Fibonacci \( p \)-number is equal to 1 identically, that is, for any integer \( i \) we have

\[
F_p(i) = 1.
\]

Then the sum (58) takes the following form:

\[
N = 1 + 1 + \cdots + 1.
\]  

(63)

Thus, the Fibonacci \( p \)-code given by (58) is a very wide generalization of the binary code (61) and Zeckendorf’s representation (62) that are the partial cases of the Fibonacci \( p \)-code (58) for the cases \( p = 0 \) and \( p = 1 \) respectively. On the other hand, the Fibonacci \( p \)-code (58) includes in itself so-called the “unitary” code (63) as another extreme case for \( p = \infty \).

9.2. Surprising analogies between Fibonacci and genetic codes

Among the biological concepts, which are well formalized and have a level of the general scientific importance, the genetic code takes a special place. Discovering the well-known fact of striking simplicity of basic principles of the genetic code falls into the major modern discoveries of human science. This simplicity consists in the fact that the inheritable information is encoded by the texts from the three-alphabetic words—triplets or codonums compounded on the basis of the alphabet that consists of the four characters being the nitrogen bases: \( A \) (adenine), \( C \) (cytosine), \( G \) (guanine), \( T \) (thiamine). The given recording system is essentially unified for all boundless set of miscellaneous alive organisms and is called the genetic code.

It is found that by using three-alphabetic triplets or codonums we can encode the 21 items including the 20 amino acids and one additional item called stop-codonum (sign of the punctuation) encoded by triplets. Then there are \( 4^3 = 64 \) different combinations from four on three nitrogen bases used for the coding the 21 items. In this connection some of the 21 items are encoded at once by several triplets. It is called as a degeneracy of the genetic code. Finding of conformity between triplets and amino acids (or signs of the punctuation) is customary treated as decryption of genetic code.

Let us consider now the 6-digit Fibonacci 1-code (Zeckendorf’s representation) that uses six Fibonacci numbers 1, 1, 2, 3, 5, 8 as digit weights:

\[
N = a_6 \times 8 + a_5 \times 5 + a_4 \times 3 + a_3 \times 2 + a_2 \times 1 + a_1 \times 1.
\]

There are the following surprising analogies between the 6-digit Fibonacci code and genetic code:

**The first analogy.** Using the 6-digit Fibonacci code we can represent the 21 integers starting from the minimum number 0 that is encoded by the 6-digit binary combination 000000 and ending with the maximum number 20 that is encoded by the 6-digit code combination 111111. Notice that using triplet’s coding we can represent also the set of 21
objects that includes the 20 amino acids and one additional object that is used for coding of the stop-codonum (the sign of the punctuation) that keeps in itself the information about termination of protein synthesis.

The second analogy. The main feature of Fibonacci code is a multiplicity of number representation. Except of the minimum number 0 and the maximum number 20, which have in Fibonacci code the only code representations (accordingly 000000 and 111111), all the rest numbers from 1 up to 19 have multiple representations in the Fibonacci code, that is, they use not less than two code combinations for their representation. It is necessary to notice that the genetic code uses the property of multiplicity representation and the latter is called degeneracy of the genetic code.

In [9,41–44] the original computer arithmetic that is based on the Fibonacci p-numbers is developed. This arithmetic can become the beginning of the noise-tolerant processors and computers [43].

9.3. Codes of the Golden proportion and “Golden” arithmetic

The codes of the Golden Proportion (32), (33) had led to the elaboration of the so-called “Golden” arithmetic [11] that is similar to the Fibonacci arithmetic and can be used for development of new computer projects.

9.4. Ternary mirror-symmetric arithmetic

The ternary mirror-symmetric arithmetic [45] is modern original computer invention. This one is a synthesis of Bergman’s number system (8) and ternary symmetrical number system used by Russian engineer and scientist Nikolay Brousentsov in “Setun” computer [46] that is the first in computer history ternary computer based on the “Brousentsov’s Ternary Principle” [45].

It is proved [45] that every natural number can be represented in the form:

\[ N = \sum_i c_i \tau^i, \]

(64)

where \( c_i \) is the ternary numeral \( \{1, 0, 1\} \) of the \( i \)th digit; \( \tau^i \) is the weight of the \( i \)th digit; \( \tau^2 \) is the radix of the number system (64). For example the ternary representation (64) of the number 5 has the following form:

\[
\begin{align*}
2 & 1 0 -1 -2 \\
5 & = 1 1 1, 1 1
\end{align*}
\]

(65)

Considering the ternary representation of the number 5 we can find that the left-hand part (11) of the ternary representation (65) is mirror-symmetric to its right-hand part (11) regarding to the 0th digit. This property of the “mirror symmetry” has a general character and is valid for all natural numbers. Taking into consideration the “mirror-symmetrical” property the ternary representation (64) is called ternary mirror-symmetrical representation [45].

It follows from (64) that the radix of the ternary mirror-symmetric number system (64) is the square of the Golden Ratio:

\[ \tau^2 = \frac{3 + \sqrt{5}}{2} \approx 2.618. \]

It means that the number system (64) is a number system with an irrational radix.

The radix of the number system (64) has the following traditional representation:

\[ \tau^2 = 10. \]

The following identities for the Golden Ratio powers underlie the ternary mirror-symmetrical addition:

\[ 2\tau^{2k} = \tau^{2(k+1)} - \tau^{2k} + \tau^{2(k-1)}; \]

(66)

\[ 3\tau^{2k} = \tau^{2(k+1)} + 0 + \tau^{2(k-1)}; \]

(67)

\[ 4\tau^{2k} = \tau^{2(k+1)} + \tau^{2k} + \tau^{2(k-1)}; \]

(68)

where \( k = 0, \pm 1, \pm 2, \pm 3, \ldots \).

The identity (66) is a mathematical basis for the mirror-symmetric addition of the two single-digit ternary digits and gives the rule of the carry formation (Table 3).

As follows from Table 3 a peculiarity of the ternary mirror-symmetric addition consists in the fact that at the addition of the significant ternary digits of the same sign (1 + 1 and 1 + 1) the carries spread symmetrically to the higher and the lower digits of the summand ternary mirror-symmetric digits.
The subtraction of the two mirror-symmetric numbers $N_1 - N_2$ is reduced to the mirror-symmetric addition if we represent their difference in the following form:

$$N_1 - N_2 = N_1 + (-N_2).$$  

(69)

It follows from (69) that before subtraction it is necessary to take the ternary inversion of the subtrahend $N_2$ according to the rule:

1 $\rightarrow \bar{1}$; 0 $\rightarrow 0$; $\bar{1} \rightarrow 1$.

The following trivial identity for the golden ratio powers underlies the mirror-symmetric multiplication:

$$\varphi^{2n} \times \varphi^{2m} = \varphi^{2(n+m)}.$$

The rule of the mirror-symmetric multiplication of the two single-digit ternary mirror-symmetric numbers is given in Table 4.

<table>
<thead>
<tr>
<th>$b_k$</th>
<th>$a_k$</th>
<th>0</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\bar{1}$</td>
<td>1</td>
<td>0</td>
<td>$\bar{1}$</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>$\bar{1}$</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

Let us multiply the negative mirror-symmetric number $-\bar{6} = \bar{1}01, 0\bar{1}$ by the positive mirror-symmetric number $2 = 11, 1$:

<table>
<thead>
<tr>
<th>$\bar{1}$</th>
<th>0</th>
<th>1,</th>
<th>0</th>
<th>$\bar{1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1,</td>
<td>$\bar{1}$</td>
<td>1</td>
<td>$\bar{1}$</td>
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<tr>
<td>1</td>
<td>0</td>
<td>1,</td>
<td>0</td>
<td>$\bar{1}$</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>$\bar{1}$</td>
<td></td>
</tr>
</tbody>
</table>

| 1 | 1 | 0 | 1, | 0 | $\bar{1}$ |

The multiplication result is formed as the sum of the three partial products. The first partial product $\bar{1}0, 10\bar{1}$ is the result of multiplication of the mirror-symmetric number $-6 = 101, 01$ by the lowest positive unit of the mirror-symmetric number $2 = 11, 1$, the second partial product $101, 01$ is the result of the multiplication of the same number $-6 = 101, 01$ by the middle negative unit of the number $2 = 11, 1$, and finally the third partial product $101, 01$ is the result of the multiplication of the same number $-6 = 1101, 01$ by the higher positive unit of the number $2 = 11, 1$. The summing up of the intermediate products is performed according to Table 3.

Note that the product $-12 = \bar{1}101, 01\bar{1}$ is represented in the mirror-symmetric form! As its higher digit is a negative unit 1 it follows from here that the product is a negative number.

Thus, the ternary mirror-symmetric arithmetic [35,37] has a number of unique mathematical properties: (1) positive and negative integers are represented in direct form; (2) all ternary mirror-symmetrical representations of integers have “mirror-symmetrical form”; (3) all arithmetical operations are performed in the “direct” form; (4) the result of every

Table 3

<table>
<thead>
<tr>
<th>$b_k$</th>
<th>$a_k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\bar{1}$</td>
<td>111</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 4

<table>
<thead>
<tr>
<th>$b_k$</th>
<th>$a_k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\bar{1}$</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>$\bar{1}$</td>
</tr>
</tbody>
</table>
arithmetical operation is represented in the “mirror-symmetrical form”. It is clear that the ternary mirror-symmetrical arithmetic can lead to the original computer projects based on the “Ternary Principle” (ternary representations of numbers, ternary logic, ternary memory element, flip-flap-flop) [45].

10. The generalized principle of the Golden Section in electrical engineering

10.1. The “binary” resistor divider

In electrical engineering practice so-called resistor dividers intended for division of electric currents and voltages in the given ratio are widely used. One variant of such divider is shown in Fig. 3.

The resistor divider in Fig. 3 consists of the “horizontal” resistors of the kind \( R_1 \) and \( R_3 \) and the “vertical” resistors of the kind \( R_2 \). The resistors of the divider are connected between themselves by the “connecting points” 0, 1, 2, 3, 4. Each point connects three resistors that form together the resistor section. Note that Fig. 3 demonstrates the resistor divider that consists of the 5 resistor sections. In general, a number of resistor sections can be equal to \( n (n = 1, 2, 3, \ldots) \).

First of all, we note that the parallel connection of the resistors \( R_2 \) and \( R_3 \) to the right of the “connecting point” 0 and to the left of the “connecting point” 4 can be replaced by the equivalent resistor with the resistance, which can be calculated according to the law on the resistor parallel connection:

\[
R_{eq1} = \frac{R_2 \times R_3}{R_2 + R_3}.
\]

(70)

Taking into consideration (70) it is easy to find the equivalent resistance of the resistor section to the right of the “connecting point” 1 and to the left of the “connecting point” 3:

\[
R_{eq2} = R_1 + R_{eq1}.
\]

(71)

In dependence on the choice of the resistance values of the resistors \( R_1, R_2, R_3 \) we can realize the different coefficients of the current or voltage division. Let us consider the so-called “binary” divider that consists of the following resistors: \( R_1 = R; R_2 = R_3 = 2R \), where \( R \) is some standard resistance value. For this case the expressions (70), (71) take the following values:

\[
R_{eq1} = R; \quad R_{eq2} = 2R.
\]

(72)

Then, taking into consideration (70)–(72) we can find that the equivalent resistance of the resistor circuit to the left or to the right of any “connecting point” 0, 1, 2, 3, 4 is equal to \( 2R \). It means that the equivalent resistance of the divider in any of the “connecting points” 0, 1, 2, 3, 4 can be calculated as the resistance of the parallel connection of the three resistors of the value \( 2R \). Using the electrical circuit laws we can calculate the equivalent resistance of the divider in each “connecting point” 0, 1, 2, 3, 4

\[
R_{eq3} = \frac{2}{3} R.
\]

(73)
Let us connect now the generator of electric current $I$ to one of the “connecting points”, for example, to the point 2. Then according to Ohm’s law the following electric voltage will appear in this point:

$$U = \frac{2}{3}RI.$$  \hfill (74)

Let us calculate the electrical voltages in the “connecting points” 3 and 1 that are adjacent to the point 2. It is easy to show that the voltage transmission coefficient between the adjacent “connecting points” is equal to $\frac{1}{4}$. It means that the “binary” divider fits very well to the binary number system and this fact is a reason of wide use of the “binary” resistor divider in modern digit-to-analog and analog-to-digit converters and modern measurement systems.

10.2. The “golden” resistor dividers

Let us take the values of the resistors in Fig. 3 as the following:

$$R_1 = \tau_p^{-1}R; \quad R_2 = \tau_p^{d+1}R; \quad R_3 = \tau_p R,$$

where $\tau_p$ is the Golden $p$-Ratio, $p \in \{0, 1, 2, 3, \ldots\}$.

It is clear that the divider in Fig. 3 gives an infinite number of the different resistor dividers as each value of $p$ “generates” a new divider. In particular, for the case $p = 0$ we have: $\tau_0 = 2$ and the divider is reduced to the classical “binary” divider.

For the case $p = 1$ the resistors $R_1, R_2, R_3$ take the following values:

$$R_1 = \tau^{-1}R; \quad R_2 = \tau^2R; \quad R_3 = \tau R,$$

where $\tau = \frac{1 + \sqrt{5}}{2}$ is the Golden Ratio.

And now we will investigate the basic electrical properties of the “golden” resistor divider (Fig. 3) given by (76). To this end we will use the following properties of the Golden $p$-Ratio:

$$\tau_p = 1 + \tau_p^{p};$$ \hfill (77)

$$\tau_p^{p+2} = \tau_p^{p+1} + \tau_p.$$ \hfill (78)

Let us calculate the equivalent resistance of the resistive circuit of the divider to the left and to the right from the “connecting points” 0 and 4 using the expression (78):

$$R_{e1} = \frac{R_2 \times R_3}{R_2 + R_3} = \frac{\tau_p^{p+1}R \times \tau_p R}{\tau_p^{p+1}R + \tau_p R} = R.$$ \hfill (79)

Note that we simplified the expression (79) using the mathematical identity (78).

Using (71) and (77) it is possible to calculate the equivalent resistance of $R_{e2}$:

$$R_{e2} = \tau_p^{-p}R + R = \tau_p R.$$ \hfill (80)

Thus, according to (80) the equivalent resistance of the resistor circuit of the divider to the left or to the right of any of the “connecting points” 0, 1, 2, 3, 4 is equal to $\tau_p R$ where $\tau_p$ is the Golden $p$-Ratio. This fact can be used for calculation of the equivalent resistance $R_{e3}$ of the divider in the “connecting points” 0, 1, 2, 3, 4. In fact, the equivalent resistance $R_{e3}$ can be calculated as the resistance of the electrical circuit that consists of the parallel connection of the “vertical” resistor $R_2 = \tau_p^{p+1}R$ and the two “lateral” resistors with the resistance $\tau_p R$. But as the equivalent resistance of the parallel connection of the resistors $R_2 = \tau_p^{p+1}R$ and $R_3 = \tau_p R$ is equal to $R$ according to (79) then the equivalent resistance $R_{e3}$ of the divider in any of the “connecting points” can be calculated by the formula:

$$R_{e3} = \frac{\tau_p R \times R}{\tau_p R + R} = \frac{\tau_p}{\tau_p + 1} R = \frac{1}{1 + \tau_p^{-1}} R.$$ \hfill (81)

Note that for the case $p = 0$ (the “binary” divider) we have: $\tau_p = \tau_0 = 2$ and then the expression (81) is reduced to (74).

Let us calculate the voltage transmission coefficient between the adjacent “connecting points” of the “golden” divider. To this end we will connect the generator of the electric current $I$ to one of the “connecting points”, for example, to the point 2. Then according to Ohm’s law the following electrical voltage appears in this point

$$U = \frac{1}{1 + \tau_p^{-1}} RI.$$ \hfill (82)
Let us calculate the electrical voltage in the adjacent “connecting points” 3 and 1. The voltages in the points 3 and 1 can be calculated as a result of linking the voltage $U$ given by (82) to the resistor circuit that consists of the sequential connection of the “horizontal” resistor $R I = \tau_p R$ and the resistor circuit with the equivalent resistance $R$. Then, for this case the electrical current that appears in the resistor circuit to the left and to the right of the “connecting point” 2 will be equal to

$$\frac{U}{R I + R} = \frac{U}{(\tau_p R + 1)R} = \frac{U}{\tau_p R}. \quad (83)$$

If we multiply the electrical current given by (83) by the equivalent resistance $R$ we will obtain the following value of the electrical voltage in the adjacent “connecting points” 3 and 1:

$$\frac{U}{\tau_p}. \quad (84)$$

It means that the voltage transmission coefficient between the adjacent “connecting points” of the “golden” divider in Fig. 3 is equal to the reciprocal of the Golden $p$-Ratio!

Thus, the “golden” resistor dividers based on the Golden $p$-Ratios $\tau_p$ are quite real electrical circuits. It is clear that the stated above theory of the “golden” resistor dividers could become a new source for development of so-called “digital metrology” and analog-to-digit and digit-to-analog converters. It is important to emphasize that new class of resistor dividers are based on the Generalized Principle of the Golden Section!

10.3. The “golden” digit-to-analog and analog-to-digit converters

The electrical circuit of the «golden» DAC based on the “golden” resistor divider in Fig. 3 is shown in Fig. 4. Note that the “golden” DAC in Fig. 4 consists of the five resistor sections. However the number of the DAC resistor sections may be increased to some arbitrary $n$ by extending the resistor divider to the left and to the right.

The “golden” DAC contains the five ($n$ in the general case) generators of the standard electrical current $I_0$ and the five ($n$ in the general case) electrical current keys $K_0$ to $K_4$. The key states are controlled by the binary digits of the Golden $p$-Ratio code $a_d a_{d-1} a_{d-2} a_1 a_0$. For the case $a_i = 1$ the key $K_i$ is closed, for the case $a_i = 0$ it is open ($i = 0, 1, 2, \ldots n$).

One can show that the closed key $K_i$ results in the following voltage at the $i$th point of the resistor divider:

$$U_i = \beta_p I_0 R,$$

where

$$\beta_p = \frac{1}{1 + \tau_p^{-1}}.$$
As the potential $U_i$ is passed from the $i$th point to the $(i + 1)$th point with the transmission coefficient $\frac{1}{\tau_p}$ the following voltage appears at the DAC output:

$$U_{out} = \frac{\beta_p I_0 R}{\tau_{p-1}} = \frac{\beta_p I_0 R}{\tau_{p-1}} \times \tau'_p.$$

Using the superposition principle it is easy to show that the Golden $p$-Ratio code $a_{n-1}a_{n-2} \ldots a_0$ results in the following voltage $U_{out}$:

$$U = B_p \sum_{i=0}^{n-1} a_i \tau'_p,$$  \hspace{1cm} (85)

where

$$B_p = \frac{\beta_p I_0 R}{\tau'_p}.$$

It follows from (85) that the electrical circuit in the Fig. 4 converts the Golden $p$-Ratio code (31) into the electrical voltage $U_{out}$ with regard to the constant coefficient $B_p$.

The “golden” DAC in Fig. 4 is a basis of so-called self-correcting ADC’s [7,16] that are “insensitive” to the DAC “technological” and “temperature” errors.

Note that the considered above “golden” ADC and DAC are the basis of new projects in the field of “digital metrology” and measurement systems.

11. The generalized principle of the Golden Section in communication systems

11.1. A new approach to Shannon’s theory of communication

Let us demonstrate application of the Generalized Principle of the Golden Section to Shannon’s communication theory. Let us consider the traditional “channel without noise” (Fig. 5).

As is well known, a channel capacity is the most important characteristic of the channel in Fig. 5. According to the communication theory developed by the American scientist Claude Shannon the channel capacity is given by the following expression:

$$C = \sup R = \lim_{T \to \infty} \frac{1}{T} I(\xi, \eta),$$  \hspace{1cm} (86)

where $R$ is a speed of information transmission in the channel, $T$ is a time of information transmission, $I(\xi, \eta)$ is average information quantity transmitted through the channel during the time $T$, $\xi$ and $\eta$ are input and output messages accordingly.

Let us consider so-called “symmetric” channel. In it the information is transmitted by bits 0 and 1 having equal length $\Delta T$. To calculate the channel capacity without the formula (86), it is necessary to know the transmission time $T$ and to calculate the information quantity $I(\xi, \eta)$. If we know a number $n$ of bits of the given message we can calculate the time of $T$ under the following trivial formula:

$$T = n \Delta T.$$  \hspace{1cm} (87)

To calculate $I(\xi, \eta)$ we have to remind that for the “channel without noise” we have:

$$I(\xi, \eta) = \log_2 N(T),$$  \hspace{1cm} (88)

where $N(T)$ is a number of all possible discrete messages that could be transmitted through the channel during the time $T$. Then taking into consideration (88) we can write the expression (86) as follows:

$$C = \lim_{T \to \infty} \frac{\log_2 N(T)}{T}.$$  \hspace{1cm} (89)

Fig. 5. The communication channel without noise.
For the "symmetric" channel without noise the number of \( N(T) \) coincides with the number of all possible \( n \)-bit messages, that is
\[
N(T) = 2^n .
\] (90)

Then using (90) we can calculate the information quantity:
\[
I(\xi, \eta) = \log_2 2^n = n .
\] (91)

Substituting (87) and (91) into (89) we can obtain the well-known Shannon’s formula for the capacity of the "channel without noise":
\[
C = \frac{1}{\Delta T} .
\] (92)

Let us consider so-called "asymmetric" channel when the bits 0 and 1 demand on the different time intervals for their transmission. Let us suppose, for example, that the bit 1 demands on its transmission the time interval \( \Delta T_1 = (p+1)\Delta T \), where \( \Delta T \) is the time interval necessary for transmission of the bit 0, \( p = 0, 1, 2, 3, \ldots \) Note that for \( p = 0 \) we have the case of the "symmetric" channel, for the rest values of \( p \) we have the cases of the "asymmetric" channels.

It is proved [47] that for the given \( p \) the channel capacity of the channel without noise is given by the following expression:
\[
C = \frac{1}{\Delta T} \log_2 \tau_p ,
\] (93)

where \( \tau_p \) is the Golden \( p \)-Ratio.

Note that for the case \( p = 0 \) (the "symmetric" channel) we have: \( \tau_p = 2 \) and the formula (93) is reduced to the classical Shannon’s formula (92).

It is clear that the above scientific result is an "elementary draft" of possible development of Shannon’s theory of communication for the "asymmetric" channel. However, for our theory it is important the following aspect of the "golden" approach to Shannon’s communication theory. As soon as we introduce the idea of "asymmetry" into consideration at once we will come to the Fibonacci \( p \)-numbers and to the Generalized Principle of the Golden Section!

11.2. Coding theory based on the Fibonacci matrices

In [47] it is developed the following coding/decoding method based on the Fibonacci matrices (41). Let us represent the initial message in the form of the square \( (p+1) \times (p+1) \)-matrix \( M \) and use the following coding/decoding method given in Table 5. We will use the direct Fibonacci \( Q_p \)-matrix \( Q_p^n \) of the same size for the coding and the inverse matrix \( Q_{0, n}^p \) for the decoding.

It follows from Table 5 that for the given \( p(p = 0, 1, 2, 3, \ldots) \) the Fibonacci coding consists in multiplication of the initial \( (p+1) \times (p+1) \)-matrix \( M \) by the coding matrix \( Q_p^n \) given by (41). The code matrix \( E \) is the result of such matrix multiplication. Then the Fibonacci decoding consists in multiplication of the code matrix \( E \) by the "inverse" matrix \( Q_{0, n}^p \). Note that the coding/decoding method is trivial for the case \( p = 0 \). However, the coding/decoding methods corresponding to the cases \( p > 0 \) represent by themselves some code transformations of the initial message \( M \) to the code matrix \( E \) and this transformation is useful for some applications.

Notice that the coding/decoding method given by Table 5 for the cases \( p > 0 \) ensure infinite variants of the Fibonacci coding/decoding as every Fibonacci coding matrix \( Q_p^n \) and its inverse matrix \( Q_{0, n}^p \) (\( p = 1, 2, 3, \ldots ; n = \pm 1, \pm 2, \pm 3, \ldots \)) "generate" their own Fibonacci coding/decoding method.

Notice that the Fibonacci coding/decoding method given by Table 5 has an unique mathematical property. Let us consider the code matrix \( E \) given by the formula
\[
E = M \times Q_p^n .
\] (94)
Let us calculate the determinant of the code matrix $E$

$$\text{Det } E = \text{Det } [M \times Q^p] = \text{Det } M \times \text{Det } Q^p. \quad (95)$$

Using (42) we can write the formula (95) as follows:

$$\text{Det } E = \text{Det } M \times (-1)^p. \quad (96)$$

Let us formulate this property as the following theorem.

**Theorem 5.** The determinant of the code matrix $E$ obtained as result of the multiplication of the initial matrix $M$ by the coding matrix $Q^p$ given by (41) is determined by the determinant of the initial matrix $M$; here they differ only by sign: if the product of $p \times n$ is even the determinants of the initial matrix $M$ and the code matrix $E$ coincide, in opposite case they differ by sign.

It is clear that the identity (96) can be considered as the main “checking relation” of the Fibonacci coding/decoding method. According to (96) we can compare the determinant of the code message $E$ to the determinant of the initial message $M$ to check correctness of the code matrix transmission through the “channel”. As is shown in [47] the Fibonacci coding/decoding method given by Table 5 can be used effectively for detection and correction of errors that can arise in communication channels.

### 11.3. Coding theory based on the “Golden” matrices

Above we have introduced the “Golden” matrices (53), (54) based on the hyperbolic Fibonacci functions (44), (45). One may suggest the following coding/decoding method based on the “Golden” matrices:

<table>
<thead>
<tr>
<th>Coding</th>
<th>Decoding</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A \times Q^{2x} = E(x)$</td>
<td>$E(x) \times Q^{-2x} = A$</td>
</tr>
<tr>
<td>$A \times Q^{2x+1} = E(x)$</td>
<td>$E(x) \times Q^{-2x+1} = A$</td>
</tr>
</tbody>
</table>

Here $A$ is the initial matrix; $E(x)$ is the code matrix; $Q^{2x}, Q^{2x+1}$ are the coding matrices that are the “Golden” matrices of the kind (53), (54) respectively.

We can use the continues variable $x$ as “cryptography key”. It means that in dependence on $x$ there is an infinite variants of transformation of the initial matrix $A$ into the coding matrix $E(x)$. By essence, the coding/decoding method given by Table 6 can be considered as a new cryptography method that can be used for security of digital signals.

### 12. The “Golden” projects—conclusion

The main result of the present article is a mathematical substantiation of a new scientific principle, the Generalized Principle of the Golden Section, that includes the “Dichotomy Principle” and “Golden Section Principle” as special cases. This principle is based on the Golden $p$-Proportions that are a new class of irrational numbers that reflect deep mathematical regularities of the Pascal triangle. This is the basis of a new mathematical direction called Harmony Mathematics. New scientific principles and Harmony Mathematics can influence on different fields of modern culture and become the source of new scientific, engineering and cultural deriving force, such as:

1. **Probability theory.** It is impossible to overestimate the methodological importance of the deep mathematical connection of the Generalized Principle of the Golden Section to the Pascal triangle and binomial coefficients. It is clear that this connection can become the beginning of a reappraisal of many branches of modern mathematics and theoretical physics where combinatorial relations play an important role, in particular, in probability theory and statistical laws.

2. **Number theory.** A new definition of real number given by (31) has “strategic” importance for the development of a number theory [14]. According to (31) there are an infinite number of “constructive” definitions of real number based on the Golden $p$-Proportions. Every definition (31) “generates” some original number theory. $Z$- and $Z_p$- properties are new mathematical properties of natural numbers that follow from these definitions.
3. Measurement theory. As is well known, measurement theory based on Eudoxus–Archimedes’ and Cantor’s axioms is one of the fundamental theories of mathematics. From this point of view algorithmic measurement theory [9,10,12] has fundamental interest for mathematics development. The Italian mathematician Leonardo Pisano (Fibonacci) became famous for two mathematical discoveries, viz. the problem of choosing the best system of the standard weights (Bachet–Mendeleev’s problem), when he invented the binary number system, and the problem of rabbits reproduction, which gave the Fibonacci numbers. The “Asymmetry Principle of Measurement”, which combined both Fibonacci’s problems, showed the deep inner connection between the “rabbits reproduction” and Bachet–Mendeleev’s problems. This analogy has fundamental interest for applications of algorithmic measurement theory, in particular for the mathematical theory of biological populations.

4. Theory of elementary functions. The hyperbolic Fibonacci and Lucas functions [38–40] are a new class of elementary functions that have “strategic” importance for the development of modern mathematics and physics. One may assume the origin of the following cosmologic theories from this approach: (1) Lobatchevski–Fibonacci geometry as the Fibonacci interpretation of Lobatchevski geometry; (2) Minkovski–Fibonacci geometry as the Fibonacci interpretation of Einstein’s theory of relativity.

5. Fibonacci numbers theory. Harmony mathematics creates new stimulus for development of the Fibonacci number theory [1–3]. First of all new recurrence relations of the algorithmic measurement theory “generate” new numerical sequences that expand the range of Fibonacci’s research. On the other hand, hyperbolic Fibonacci and Lucas functions [38–40] transform the Fibonacci numbers theory into “continued” theory that allows to apply to the Fibonacci numbers theory mathematical apparatus of the “continued” mathematics, in particular differentiation and integration.

6. Matrix theory. The Fibonacci matrices (41) and the “golden” matrices of the kind (53), (54) have unique mathematical properties (42), (56), (57). Studying these matrices is an interesting direction in matrix theory. A search of applications of these matrices in theoretical physics is one of the important directions of the physics research.

7. Physics. The articles [19–32] exhibit a substantial interest of modern theoretical physics and the physics of high energy particles in the Golden Section. The works of Shechtman, Butusov, Mauldin and William, El Naschie, and Vladimirov show that it is impossible to imagine the future progress in physical and cosmological research without the Golden Section. The famous Russian physicist-theoretician Prof. Vladimirov (Moscow University) finished his book “Metaphysics” [29] with the following words: “Thus, it is possible to assert that in the theory of electroweak interactions there are relations that are approximately coincident with the “Golden Section” that play an important role in the various areas of science and art”.


10. Biology. Cell growth is one of the actual biological problems. It was proven in [34] that binary cell division is regularly asymmetric and based on the Fibonacci p-numbers that follows from the Pascal triangle. It means that the binary cell division is based on the Generalized Principle of the Golden Section.

11. Medicine. The Russian biologist Viktor Tsvetkov asserts in his book “Heart, the Golden Section and Symmetry” [48] that mammal’s cardiogram is subordinated to the Golden Section Principle. In Tsvetkov’s opinion, the organization of the “golden” cardiac cycle is a result of a long evolution of mammals in the direction of optimization of their structure and functions.

12. Computers. Fibonacci and “golden” computer arithmetic that was developed in [9,11,41–45] can become the basis of new computer projects (Fibonacci noise-tolerant processor [43], “Golden” fault-tolerant ternary mirror-symmetrical processor [45] and so on).

13. Measurement systems. The “golden” resistor dividers are the electrical basis of the “golden” analog-to-digit and digit-to-analog converters [43] and could lead to new measurement system projects.

14. Communication systems. Fibonacci matrices (41) and the “golden” matrices (53), (54) are the mathematical basis for new a coding theory [47] that could be used effectively for error detection and correction in communication channels and for cryptographic protection of communication systems.

15. Museum of Harmony and the Golden Section [49,50] is a unique History, Science, Art and Nature museum that has no analogy in world culture. The Museum is a collection of all Science, Art and Natural works based on the Golden Section.

16. Reform of mathematical education. Traditionally so-called “Elementary Mathematics” that was developed in ancient time underlies the basis of mathematical education in secondary schools. Johannes Kepler once said:
"There are two treasures in Geometry: Pythagorean Theorem and a division of line in extreme and mean ratio ('Golden Section'). The former can be compared to gold value; the latter can be named as a gemstone". Every student knows Pythagorean Theorem, but sometimes many so-called "educated people" have a dim knowledge about the Golden Section. Harmony Mathematics is the development and supplement of the "Elementary Mathematics" and can become the basis of a mathematical education reform based on the Golden Section. Mathematics of Harmony can transform process of mathematics studying into fascinating search of mathematical laws in the world, which surrounds us.


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