The Golden Shofar
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Abstract

The goal of the present article is to develop the “continues” approach to the recurrent Fibonacci sequence. The main result of the article is new mathematical model of a curve-linear space based on a special second-degree function named “The Golden Shofar”.

1. Introduction

As it is well known that the concept of Harmony, which is based on the Golden Section, is one of the most important scientific concepts of ancient science. In the Ancient Greek’s philosophy, Harmony was opposed to Chaos and meant the organization of the Universe. The ingenious Russian philosopher Alexey Losev writes: “From Plato’s point of view and generally from the view of all antique cosmology, the Universe by itself is the certain proportional whole that subordinates to the law of harmonic division, namely, the Golden Section”.

The concepts of Symmetry and Harmony are used widely in modern physical research[1–16]. In the paper [3] written by Mauldin and Williams in 1986 they “proved a theorem which at first sight may seem slightly paradoxical but we perceive as excitingly interesting. This theorem states that the Hausdorf dimension $d_0^{(0)}$ of a randomly constructed Cantor set is $d_0^{(0)} = \phi$, where $\phi = \sqrt{5} - 1 \over 2$ is the Golden Mean” (quotation from [6]). The well-known Russian theoretical physicist Prof. Vladimirov (Moscow University) finished his book on “Metaphysics” [16] with the following words: “Thus, it is possible to assert that in the theory of electroweak interactions there are relations that approximately coincide with the “Golden Section” that play an important role in the various areas of science and art”.

The works of Shechtman, Butusov, Mauldin and William, El Naschie, and Vladimirov show that it is not easily possible to imagine the future progress in physical and cosmological research without the Golden Section.

That is why the development of the new mathematical apparatus based on the Golden Section is an important problem of modern theoretical physics. The present article is continuation of the works [17–19] that were devoted to the theory of the hyperbolic Fibonacci and Lucas functions based on the Golden Section. The main result of the article is a new mathematical model of a curve-linear space based on a special second-degree function named “The Golden Shofar.”

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2. Symmetric hyperbolic Fibonacci functions

Let us consider Binet’s formula for Fibonacci numbers

\[ F_n = \frac{\phi^n - (-1)^n \phi^{-n}}{\sqrt{5}}, \]

(1)

where \( \phi = \frac{1 + \sqrt{5}}{2} \) is the “golden proportion” and \( n = 0, \pm 1, \pm 2, \pm 3, \ldots \)

Binet’s formula (1) is the basis for the introduction of the hyperbolic Fibonacci and Lucas functions [17,18]. Developing the approach stated in [17,18] and using (1) the symmetric hyperbolic Fibonacci functions, which were developed in the article [19], namely:

**Symmetric hyperbolic Fibonacci sine**:

\[ sFs(x) = \frac{\phi^x - \phi^{-x}}{\sqrt{5}}. \]

(2)

**Symmetric hyperbolic Fibonacci cosine**:

\[ cFs(x) = \frac{\phi^x + \phi^{-x}}{\sqrt{5}}. \]

(3)

The graphs of the symmetric hyperbolic sine and cosine are shown in Fig. 1.

3. The quasi-sine Fibonacci function

Comparing Binet’s formulas (1) for the Fibonacci numbers to the symmetric hyperbolic Fibonacci functions (2) and (3), it is possible to see that the continuous functions \( \phi^x \) and \( \phi^{-x} \) in the formulas (2) and (3) correspond to the discrete sequences \( \phi^n \) and \( \phi^{-n} \) in the formula (1). Consequently it is possible to insert some continuous function that takes the values \(-1\) and \(1\) in the discrete points \( x = 0, \pm 1, \pm 2, \pm 3, \ldots \) that corresponds to the alternating sequence \((-1)^n\) in Binet’s formula (1). The trigonometric function \( \cos(\pi x) \) is the simplest of them all. This reasoning is the basis for the introduction of new continuous functions that are connected to the Fibonacci numbers.

**Definition 1.** The following continuous function is called the quasi-sine Fibonacci function:

\[ FF(x) = \frac{\phi^x - \cos(\pi x)\phi^{-x}}{\sqrt{5}}. \]

(4)
There is the following correlation between Fibonacci numbers \( F_n \) given by (1) and quasi-sine Fibonacci function given by (4):

\[
F_n = FF(n) = \frac{x^n - \cos(\pi n)x^{-n}}{\sqrt{5}},
\]

where \( n = 0, \pm 1, \pm 2, \pm 3, \ldots \)

The graph of the QSFF is the quasi-sine curve that passes through all points corresponding to the Fibonacci numbers given by (5) on the coordinate plane (Fig. 2). The symmetric hyperbolic Fibonacci functions (2) and (3) (Fig. 1) are the envelopes of the quasi-sine Fibonacci function.

**Theorem 1.** For the quasi-sine Fibonacci function there is the following correlation similar to the recurrence relation for the Fibonacci numbers \( F_{n+2} = F_{n+1} + F_n \):

\[
FF(x + 2) = FF(x + 1) + FF(x).
\]

**Proof**

\[
\begin{align*}
FF(x + 1) + FF(x) &= \frac{x^{x+1} - \cos(\pi(x + 1))x^{-x-1}}{\sqrt{5}} + \frac{x^x - \cos(\pi x)x^{-x}}{\sqrt{5}}
= \frac{x^{x+1} + \cos(\pi)ax^{-x-1}}{\sqrt{5}} + \frac{x^x - \cos(\pi x)x^{-x}}{\sqrt{5}} \\
&= \frac{x^{x+1} + \cos(\pi(x + 2))x^{-x}}{\sqrt{5}} \\
&= FF(x + 2). \quad \Box
\end{align*}
\]

**Theorem 2.** For the quasi-sine Fibonacci function there is the following correlation similar to the following formula \( F_{n+2}F_{n-1} - F_{n+1}F_n = (-1)^{n+1} \):

\[
[FF(x)]^2 - FF(x + 1)FF(x - 1) = -\cos(\pi x).
\]
Proof

\[
[FF(x)]^2 - FF(x+1)FF(x-1) = \left(\frac{a^x - \cos(\pi x) a^{-x}}{\sqrt{5}}\right)^2 - \frac{a^{x+1} - \cos(\pi(x+1)) a^{-x-1}}{\sqrt{5}} \times \frac{a^{x-1} - \cos(\pi(x-1)) a^{x+1}}{\sqrt{5}}
\]

\[
= \frac{a^{2x} - 2a^x \cos(\pi x) + [\cos(\pi x)]^2 a^{-2x} - (a^x + \cos(\pi x)) a^x + \cos(\pi x) a^{-1} + [\cos(\pi x)]^2 a^{-x}}{\sqrt{5}}
\]

\[
= \frac{-\cos(\pi x)(2 + a^x + a^{-x})}{5} = -\cos(\pi x). \quad \square
\]

By analogy with Theorems 1 and 2 one may prove other identities for the quasi-sine Fibonacci function (see below).

<table>
<thead>
<tr>
<th>The identities for Fibonacci numbers</th>
<th>The identities for the quasi-sine Fibonacci function</th>
</tr>
</thead>
<tbody>
<tr>
<td>(F_{n+3} + F_n = 2F_{n+2})</td>
<td>(FF(x+3) + FF(x) = 2FF(x+2))</td>
</tr>
<tr>
<td>(F_{n+3} - F_n = 2F_{n+1})</td>
<td>(FF(x+3) - FF(x) = 2FF(x+1))</td>
</tr>
<tr>
<td>(F_{n+6} - F_n = 4F_{n+3})</td>
<td>(FF(x+6) + FF(x) = 4FF(x+3))</td>
</tr>
<tr>
<td>(F_n^2 - F_{n+1}F_{n-1} = (-1)^{n+1})</td>
<td>([FF(x)]^2 - FF(x+1)FF(x-1) = -\cos(\pi x))</td>
</tr>
<tr>
<td>(F_{2n+1} = F_{n+1}^2 + F_n^2)</td>
<td>(FF(2n+1) = (FF(n+1))^2 + (FF(n))^2)</td>
</tr>
</tbody>
</table>

4. Three-dimensional Fibonacci spiral

It is well known that the trigonometric sine and cosine can be defined as projection of the translational movement of a point on the surface of an infinite rotating cylinder with the radius 1 and the symmetry center that coincides with the axis \(OX\). Such three-dimensional spiral is projected into the sine function on a plane and described by the complex function \(f(x) = \cos(x) + i \sin(x)\), where \(i = \sqrt{-1}\).

If we assume that the quasi-sine Fibonacci function (4) is a projection of the three-dimensional spiral that is on some funnel-shaped surface and conduct reasoning similar for the reasoning for the trigonometric sine it is possible to construct so-called “three-dimensional Fibonacci spiral”.

Fig. 3. The three-dimensional Fibonacci spiral.
\textbf{Definition 2.} The following function is called the three-dimensional Fibonacci spiral:
\[
CFF(x) = \frac{x^x - \cos(\pi x)x^{-x}}{\sqrt{5}} + \frac{i \sin(\pi x)x^{-x}}{\sqrt{5}}.
\] (6)

This function, by its shape, reminds a spiral that is drawn on the crater with the bent end (Fig. 3).

\textbf{Theorem 3.} For the three-dimensional Fibonacci spiral the following correlation that is similar to the recurrence relation for Fibonacci numbers \(F_{n+2} = F_{n+1} + F_n\) is valid:
\[
CFF(x + 2) = CFF(x + 1) + CFF(x).
\]

\textbf{Proof}
\[
CFF(x + 1) + CFF(x) = \frac{x^{x+1} - \cos(\pi(x + 1))x^{-(x+1)}}{\sqrt{5}} + \frac{i \sin(\pi(x + 1))x^{-(x+1)}}{\sqrt{5}} + \frac{x^x - \cos(\pi x)x^{-x}}{\sqrt{5}} + \frac{i \sin(\pi x)x^{-x}}{\sqrt{5}}
\]
\[
= \frac{x^x(x + 1) + \cos(\pi x + 2\pi)x^{-(x+1)}(1 - x)}{\sqrt{5}} + \frac{i \sin(\pi x + 2\pi)x^{-(x+1)}(1 - 2x)}{\sqrt{5}}
\]
\[
= \frac{x^2 - \cos(\pi(x + 2))x^{-2}x^{-2}}{\sqrt{5}} + \frac{i \sin(\pi(x + 2))x^{-2}x^{-2}}{\sqrt{5}} = CFF(x + 2).
\]

\section{5. The Golden Shofar}

It is possible to select the real and imaginary parts in the three-dimensional Fibonacci spiral (6):
\[
\text{Re}(CFF(x)) = \frac{x^x - \cos(\pi x)x^{-x}}{\sqrt{5}},
\] (7)
\[
\text{Im}(CFF(x)) = \frac{\sin(\pi x)x^{-x}}{\sqrt{5}}.
\] (8)

The following system of equations is gotten from (6)–(8) if we consider the axis \(OY\) as the real axis and the axis \(OZ\) as the imaginary axis:
\[
\begin{align*}
\begin{cases}
y(x) &= \frac{x^x - \cos(\pi x)x^{-x}}{\sqrt{5}}, \\
z(x) &= \frac{\sin(\pi x)x^{-x}}{\sqrt{5}}.
\end{cases}
\end{align*}
\] (9)

Let us square both expressions of the equations system (9) and add them. Taking \(y\) and \(z\) as independent variables, there is gotten the curvilinear surface of the second degree called the Golden Shofar.

\textbf{Definition 3.} The following curvilinear function of the second degree is called the Golden Shofar:
\[
\left( y - \frac{x^x}{\sqrt{5}} \right)^2 + z^2 = \left( \frac{x^{-x}}{\sqrt{5}} \right)^2.
\] (10)

The obtained three-dimensional surface is similar to the horn or crater with the narrow bent up end (Fig. 4). Translating from the Hebrew language, the word “Shofar” means horn and is a symbol of power or might. The Shofar is blown in The Judgment Day (the Jewish New Year) and the Day of Atonement (the Yom Kippur).

The formula for the “Shofar” can be represented in the following form:
\[
z^2 = [cFs(x) - y][sFs(x) + y],
\] (11)

where \(sFs(x)\) and \(cFs(x)\) are the symmetric hyperbolic Fibonacci functions correspondingly.
Fig. 4. The Golden Shofar.

Fig. 5. The projection of the Surface Shofar on the plane \( XOY \).
The projection of the Golden Shofar on the plane $XOY$ is shown in Fig. 5. The Golden Shofar is projected into space between the graphs of the symmetric hyperbolic Fibonacci sine and cosine (Fig. 1). The three-dimensional Fibonacci spiral (6) is projected into the quasi-sine Fibonacci function (4).

The function (6) lies on the Golden Shofar and "pierces" the plane $XOY$ in the points that correspond to the terms of the Fibonacci sequence (Fig. 5).

The projection of the Golden Shofar on the plane $XOZ$ is shown in Fig. 6. The Golden Shofar is projected into space between the graph of the two exponent functions $-\frac{a^{-x}}{\sqrt{5}}$ and $\frac{a^{-x}}{\sqrt{5}}$.

Cutting the Golden Shofar by the planes that are parallel to the plane $YOZ$, the circles with the center $\left(0; \frac{x^2}{\sqrt{5}}\right)$ and the radius $\frac{a^x}{\sqrt{5}}$ will be gotten. It is possible to say that $y(x) = \frac{x^2}{\sqrt{5}}$ is the pseudo-axis of symmetry (or the axis of pseudo-symmetry) of the Golden Shofar (Fig. 4).

It is possible to regard that the Golden Shofar as a new model for a field with the curvilinear structure is similar to the model of gravitational field that is used in the general theory of relativity [20,21]. If an observer is in the area $x \ll 0$, then the distinction between the models is insignificant but as an observer moves along the axis $OX$, the curvature extent of the Golden Shofar increases. The principal distinction between the models consists in the fact that, moving to the source of a field relatively to the "Absolute system of the coordinates," an observer moves by the curve $y(x) = \frac{x^2}{\sqrt{5}}$ whereas in the "Eigen-proper system of the coordinates," he continues to move by straight line.

6. Conclusion

The authors performed their research programme in the following sequence:

- The Golden Section
- Binet’s formulas
- hyperbolic Fibonacci functions
- quasi-sine Fibonacci function
- three-dimensional Fibonacci spiral
- Golden Shofar.

The introduction of the new curve-linear space model based on the Golden Shofar is the main fundamental result of the paper. The authors hypothesize that the Golden Shofar, which follows from the Golden Section and hyperbolic Fibonacci functions, may be used widely in modern researches to model processes within curve-linear space in the context of the general theory of relativity as well as life science. It may be possibly that the Golden Shofar becomes a central...
concept for new cosmological theories based on the Golden Section and hyperbolic functions. It is possible to suppose that this approach leads to the following cosmological theories:

(1) Lobatchevski–Fibonacci geometry as Fibonacci interpretation of Lobatchevski geometry;
(2) Minkovski–Fibonacci geometry as Fibonacci interpretation of Einstein’s theory of relativity.

Only time will show if this hypothesis will turn out to be correct.

References