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THE GEOMETRY OF THE TORUS UNIVERSE

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In this contribution, we show that the cyclic universe models naturally emerge from torus geometry in a braneworld scenario. The Riemannian metric on torus and the fundamental tensors of the General Relativity are derived. A discussion on particular aspects of this model is also given.

Keywords: Cyclic universe; general relativity; torus geometry.

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1. Introduction

String theory suggests that we may live in a higher dimensional space–time[1] and the most recent cosmological model, the Ekpyrotic Universe, is based on this assumption.[2] This approach advances the idea that our universe represents a three-dimensional brane embedded in a higher dimensional bulk space–time.

Historically, the extra dimensions theory belongs to Kaluza and Klein. In 1920s Kaluza developed a unified theory of electrodynamics and gravity that required five dimensions. Later, Klein advanced the idea that the extra dimension should be very small and compactified. The modern approach of the braneworld scenario is discussed in the Randall–Sundrum work[3] which aims originally to solve the hierarchy problem. Their first model starts from M theory in the low-energy limit of 11-dimensional supergravity. If six dimensions are compactified then remains five large dimensions, as in Horava–Witten model.[4] The bulk space in Randall–Sundrum model is anti-de Sitter with negative cosmological constant. The standard model fields are open strings confined on the four-dimensional world which can exchange gravitons (closed strings) with the other brane.

Another aspect, considered in the work of Turok and Steinhardt[1] is the cyclic behavior of the brane universe. In their model, the interaction between the two branes occurs periodically, so the universe expands and contracts following an endless succession of births and crunches. Most of the predictions of the inflation cosmology are successfully fulfilled and moreover, most of the problems of the standard cosmology are avoided through this approach.

The present work shows that the cyclic universe models are obtained in a picture with toroidal space–time embedded in a five-dimensional bulk, with large extra-dimension. The torus geometry is presented and the fundamental tensors of General Relativity (GR) are computed using GR tensor package under Maple 8. In the last section it is shown that the ratio of the torus radii defines three oscillating models.

2. Torus Geometry

A torus is a ring-shaped surface generated by rotating a circle around an axis that does not intersect the circle. Alternatively, we can obtain a torus by gluing the opposite edges of a rectangle. In this model, our three-dimensional space is represented as the great circle of the torus which evolves on torus surface.

Toroidal geometry is present in the literature through the work of Arfken,[5] Moon and Spencer,[6] Morse and Feshbach.[7]

2.1. *The Riemannian metric*

Let us denote by α and β the radius of the tube and the radius from the torus center to the center of the tube and by $x_a, a = 1 \cdots 3$ the coordinates of a Cartesian system with Ox_3 axis on the direction of the torus axis of symmetry (Fig. 1).

Then, the surface equation in Cartesian coordinates is

$$\left(\beta - \sqrt{x_1^2 + x_2^2}\right)^2 + x_3^2 = \alpha^2 \tag{1}$$

and the parametric equations are

$$\begin{aligned} x_1 &= (\beta + \alpha\cos\nu)\cos\phi, \\ x_2 &= (\beta + \alpha\cos\nu)\sin\phi, \\ x_3 &= \alpha\sin\nu. \end{aligned} \tag{2}$$

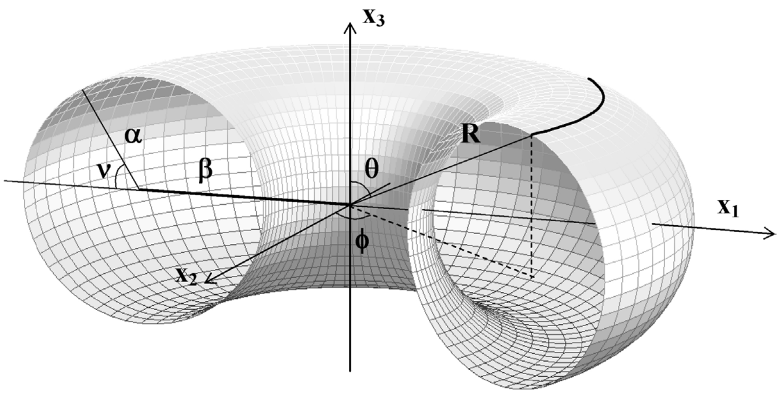


Fig. 1. Coordinate systems and parameters defined on the torus. In this representation, the time is given by the variation of angular parameter ν and the "radius" of the universe is $R\sin\theta = x_1$ (at $\phi = 0$).

These equations give the Riemannian metric

$$ds^2 = -(\beta + \alpha\cos\nu)^2 d\phi^2 + \alpha^2 d\nu^2. \tag{3}$$

Alternatively, the position of a point on the torus surface is specified in spherical coordinates by (R, θ, ϕ) and we have

$$\begin{aligned} x_1 &= R\sin\theta\cos\phi, \\ x_2 &= R\sin\theta\sin\phi, \\ x_3 &= R\cos\theta. \end{aligned} \tag{4}$$

From (2) and (4) we find

$$R^2 - 2R\beta\sin\theta + \beta^2 - \alpha^2 = 0. \tag{5}$$

Equation (5) will be used later.

2.2. *General relativity tensors*

The line element (3) is used in this section in order to obtain the GR tensors. The Maple program and the results are listed below.

```
> restart;
> with(tensor):
> coords:=[phi, nu]:
g:=array (symmetric, sparse, 1···2, 1···2):
g[1,1]:= − (beta+alpha∗cos(nu))^2: g[2,2]:=alpha^2:
metric:=create([−1,−1], eval(g));
> tensorsGR(coords,metric,contra_metric,det_met, C1, C2, Rm, Rc, R, G, C):
> display_allGR (coords,metric,contra_metric,det_met, C1, C2, Rm, Rc, R, G, C);
```

The results are listed as follows:

The Christoffel symbols of the first kind are

$$\begin{aligned} \Gamma_{11,2} &= (\beta + \alpha\cos\nu)\alpha\sin\nu, \\ \Gamma_{12,1} &= -(\beta + \alpha\cos\nu)\alpha\sin\nu, \end{aligned} \tag{6}$$

and the connection coefficients are

$$\begin{aligned} \Gamma^1_{12} &= -\frac{\alpha\sin\nu}{(\beta + \alpha\cos\nu)}, \\ \Gamma^2_{11} &= \frac{(\beta + \alpha\cos\nu)\sin\nu}{\alpha}. \end{aligned} \tag{7}$$

The Riemannian tensor non-zero component is

$$R_{1212} = \alpha\beta\cos\nu + \alpha^2\cos^2\nu. \tag{8}$$

The Ricci tensor components are

$$R_{11} = -\frac{(\beta + \alpha \cos \nu) \cos \nu}{\alpha},$$
$$R_{22} = -\frac{\alpha \cos \nu}{\beta + \alpha \cos \nu}. \tag{9}$$

The Ricci scalar is

$$Rs = -\frac{2 \cos \nu}{\alpha(\beta + \alpha \cos \nu)}. \tag{10}$$

All the Einstein tensor and Weyl tensor components are zero.

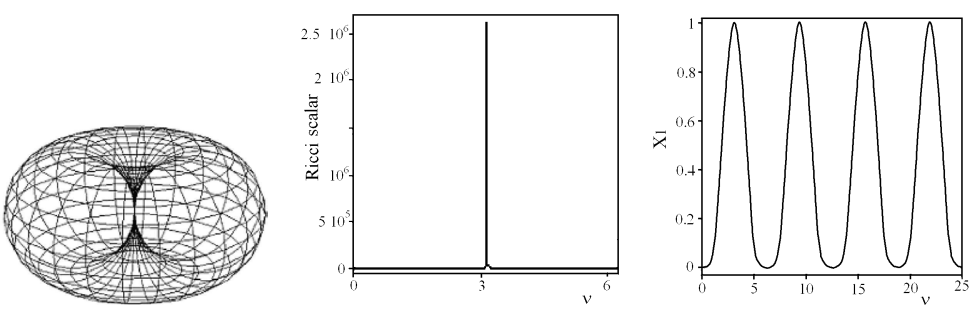
3. Cyclic Universes

Equation (5) has the solutions

$$R_{1,2} = \beta \sin \theta \pm \sqrt{\beta^2 \sin^2 \theta - (\beta^2 - \alpha^2)} \tag{11}$$

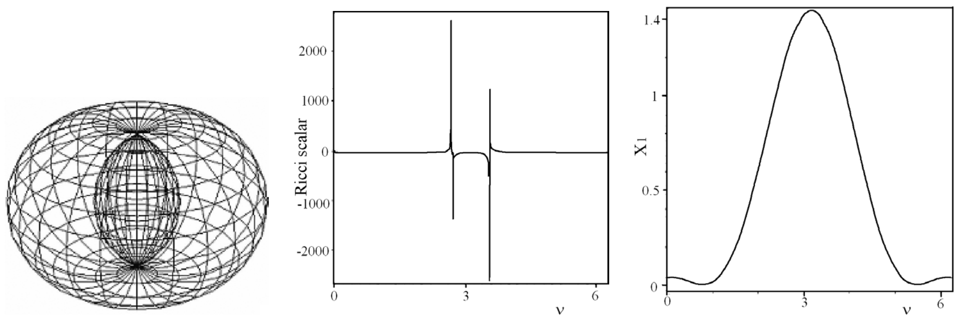
and from these solutions arise three particular cases as follows:

- $\alpha = \beta$. We have the solutions $R_1 = 0$ and $R_2 = 2\beta \sin \theta$. The latter solution is more interesting: it corresponds to a closed universe with a classical big-bang in the sense of an initial singularity, with a big crunch at $\theta = \pi$ and with a maximum of expansion at $\theta = \pi/2$, where $R = 2\beta$. A horn torus illustrates this case. (Fig. 2)
- $\alpha > \beta$. The solutions are $R_{1,2} = \beta \sin \theta \pm \beta \sqrt{(\frac{\alpha}{\beta})^2 - \cos^2 \theta}$ and describe the evolution of a universe with double period. (Fig. 3)
- $\beta > \alpha$, $R_{1,2} = \beta \sin \theta \pm \sqrt{\beta^2 \sin^2 \theta + \alpha^2 - \beta^2}$ (Fig. 4)



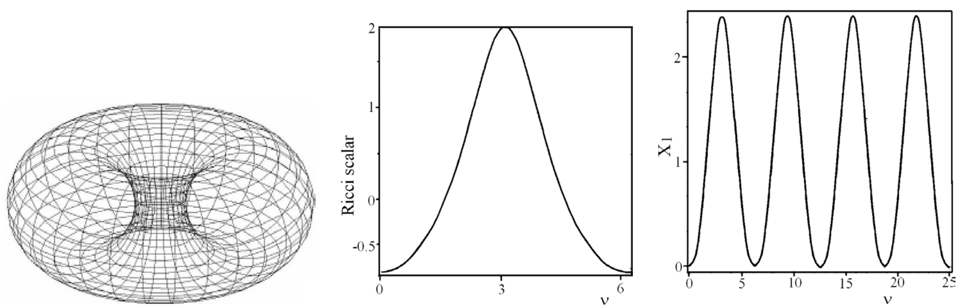
(a) (b) (c)

Fig. 2. The horn torus (a) shows an oscillating universe with Big Bang. (b) The scalar curvature is zero except the Big Bang point. (c) The coordinate x_1 is also plotted as a function of ν. In this representation the radii of the torus are equal to unity.



(a) (b) (c)

Fig. 3. The case $\alpha > \beta$ with $\alpha = 1.1$ and $\beta = 1$. The first of the two solutions is illustrated. The spindle torus (a) illustrates a two-period oscillating universe. (b) Again, the scalar curvature is zero except the two singularities. (c) The two periods, with different amplitudes, are illustrated.



(a) (b) (c)

Fig. 4. The ring shaped torus (a) represents an oscillating universe without singularities. The panels (b) and (c) show the dependence of the Ricci scalar and x_1 as functions of ν with $\alpha = 1$ and $\beta = 2$.

4. Concluding Remarks

In this paper, it has been shown that the torus geometry offers a way to approach the cyclic universe model in a simple manner. The GR tensors were derived and it was shown that three different oscillating models emerge. Using the scalar curvature expression and the coordinates on torus we can visualize the behavior of these models.

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